# The splitting lemmas for nonsmooth functionals on Hilbert spaces I*

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**Abstract**

The Gromoll-Meyer's generalized Morse lemma (so called splitting lemma) near degenerate critical points on Hilbert spaces, which is one of key results in infinite dimensional Morse theory, is usually stated for at least $C^2$-smooth functionals. It obstructs one using Morse theory to study most of variational problems of form $F(u) = \int_\Omega f(x, u, \cdots, D^m u)dx$ as in (1.1). In this paper we establish a splitting theorem and a shifting theorem for a class of continuously directional differentiable functionals (lower than $C^1$) on a Hilbert space $H$ which have higher smoothness (but lower than $C^2$) on a densely and continuously imbedded Banach space $X \subset H$ near a critical point lying in $X$. (This splitting theorem generalize almost all previous ones to my knowledge). Moreover, a new theorem of Poincaré-Hopf type and a relation between critical groups of the functional on $H$ and $X$ are given. Different from the usual implicit function theorem method and dynamical system one our proof is to combine the ideas of the Morse-Palais lemma due to Duc-Hung-Khai [19] with some techniques from [27, 43, 46]. Our theory is applicable to the Lagrangian systems on compact manifolds and boundary value problems for a large class of nonlinear higher order elliptic equations.

**Keywords.** Critical groups, splitting lemma, shifting theorem

# Contents



*IN MEMORY OF PROFESSOR SHUZHONG SHI (1939–2008)

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# 1 Introduction

## 1.1 Motivation

Morse theory is an important tool in critical point theory. Morse inequalities, which provide the appropriate relations between global topological notions and the critical groups of the critical points, had been generalized to very general frameworks, see [11, 37] (for $C^1$-functionals on manifolds of infinite dimension) and [17] (for continuous functionals on complete metric spaces) and the references therein. These inequalities and precise computations of critical groups are extremely useful in distinguishing different types of critical points and obtaining multiple critical points of a functional (cf. [4, 11, 37, 40]). However, the calculation of critical groups in applications is a complex problem. Gromoll-Meyer's generalization of Morse lemma to an isolated degenerate critical point in [23], also called the splitting theorem, provides a basic tool for the effective computation of critical groups. Since then many authors made their effort to improve the splitting theorem, see [11, 24, 37, 26, 27, 30, 19, 20, 31] and related historical and bibliographical notes in [11, Remark 5.1] and [37, page 202]. Probably, the most convenient formulations in the present applications are ones given in [10, Th. 2.1] (see also [11, Th. 5.1]) and [37, Th.8.3] (see also [36]). It was only assumed therein that $f$ is a $C^2$-functional on a neighborhood $U$ of the origin $\theta$ in a Hilbert space $H$ and that $\theta$ is an isolated critical point of $f$

such that $0$ is either an isolated point of the spectrum $\sigma(d^2f(\theta))$ or not in $\sigma(d^2f(\theta))$. This can be used to deal with many elliptic boundary value problems of form $\triangle u = f(x,u)$ on bounded smooth domains in $\mathbb{R}^n$ with Dirichlet boundary condition.

However, the action functionals in many important variational problems are at most $C^{2-0}$ on spaces where the functionals can satisfy the (PS) condition. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index of nonnegative integer components $\alpha_i$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ be its length. Denote by $M(m)$ the number of such $\alpha$ of length $|\alpha| \le m$, and by $\xi = \{\xi_\alpha : |\alpha| \le m\} \in \mathbb{R}^{M(m)}$. Consider the variational problem

$$F(u) = \int_\Omega f(x, u, \cdots, D^m u)dx, \tag{1.1}$$

where the function $f : \overline{\Omega}\times\mathbb{R}^{M(m)} \to \mathbb{R},\ (x,\xi) \mapsto f(x,\xi)$ is measurable in $x$ for all values of $\xi$, and twice continuously differentiable in $\xi$ for almost all $x$; and there are continuous, positive, nondecreasing function $g_1$ and nonincreasing function $g_2$ such that the functions

$$\overline{\Omega} \times \mathbb{R}^{M(m)} \to \mathbb{R},\ (x,\xi) \mapsto f_{\alpha\beta}(x,\xi) = \frac{\partial^2 f(x,\xi)}{\partial x_\alpha \partial x_\beta}$$

satisfy:

$$|f_{\alpha\beta}(x,\xi)| \le g_1\left(\sum_{|\gamma|<m-n/2} |\xi_\gamma|\right) \cdot \left(1 + \sum_{m-n/2\le|\gamma|\le m} |\xi_\gamma|^{p_\gamma}\right)^{p_{\alpha\beta}},$$

$$\sum_{|\alpha|=|\beta|=m} f_{\alpha\beta}(x,\xi)\eta_\alpha\eta_\beta \ge g_2\left(\sum_{|\gamma|<m-n/2} |\xi_\gamma|\right) \cdot \left(\sum_{|\alpha|=m} \eta_\alpha^2\right),$$

for any $\eta \in \mathbb{R}^{M_0}$ ( $M_0 = M(m) - M(m-1)$), where $p_\gamma$ is an arbitrary positive number if $|\gamma| = m - \frac{n}{2}$, and $p_\gamma = \frac{2n}{n-2(m-|\gamma|)}$ if $m - \frac{n}{2} < |\gamma| \le m$, and $p_{\alpha\beta} = p_{\beta\alpha}$ are defined by

$$p_{\alpha\beta} = \begin{cases} 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta} & \text{if } |\alpha| = |\beta| = m, \\ 1 - \frac{1}{p_\alpha}, & \text{if } m - \frac{n}{2} \le |\alpha| \le m,\ |\beta| < m - \frac{n}{2}, \\ 1 & \text{if } |\alpha|, |\beta| < m - \frac{n}{2}, \end{cases}$$
$$0 < p_{\alpha\beta} < 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta} \quad \text{if } |\alpha|,\ |\beta| \ge m - \frac{n}{2},\ |\alpha| + |\beta| < 2m.$$

Generally speaking, under the assumptions above, as stated on the pages 118-119 of [44] (see [43] for detailed arguments) the functional $F$ in (1.1) is $C^1$ and satisfies the (PS) condition on $W^{m,2}_0(\Omega)$, and the mapping $F'$ is only $G$-differentiable on $W^{m,2}_0(\Omega)$; moreover, on Banach spaces on $W^{m,p}_0(\Omega)$ with $p > 2$, it is $C^2$, but does not satisfy the (PS) condition. Furthermore, Morse inequalities were also obtained in [43, Chapter 5] under

the assumptions that the functional $F$ have only nondegenerate critical points. A similar question appears in some optimal control problems (see Vakhrameev [46]).

Another important problem comes from the study of periodic solutions of Lagrangian systems on compact manifolds, whose variational functional is given by

$$\mathcal{L}_\tau(\gamma) = \int_0^\tau L(t, \gamma(t), \dot\gamma(t))dt \tag{1.2}$$

on the Riemannian-Hilbert manifold $H_\tau = W^{1,2}(\mathbb{R}/\tau\mathbb{Z}, M)$ $(\subset C(\mathbb{R}/\tau\mathbb{Z}, M))$, where $M$ is a $n$-dimensional compact smooth manifold without boundary, and $L : \mathbb{R} \times TM \to \mathbb{R}$ is a $C^2$-smooth function satisfying the following conditions (L1)-(L3):

(L1) $L(t+1, q, v) = L(t, q, v) \quad \forall (t, q, v)$.

In any local coordinates $(q_1, \cdots, q_n)$, there exist constants $0 < c < C$, depending on the local coordinates, such that

(L2) $c|\mathbf{u}|^2 \le \sum_{ij} \frac{\partial^2 L}{\partial v_i \partial v_j}(t, q, v)u_i u_j \le C|\mathbf{u}|^2 \quad \forall \mathbf{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$,

(L3) $\left|\frac{\partial^2 L}{\partial q_i \partial v_j}(t, q, v)\right| \le C(1 + |v|)$ and $\left|\frac{\partial^2 L}{\partial q_i \partial q_j}(t, q, v)\right| \le C(1 + |v|^2) \quad \forall (t, q, v)$.

Under these assumptions the functional $\mathcal{L}_\tau$ is only $C^{2-0}$ on the Hilbert manifold $H_\tau$ (as showed [1] recently), but satisfies the (PS) condition on $H_\tau$. The usual regularity theory shows that all critical points of $\mathcal{L}_\tau$ on $H_\tau$ sit in the Banach manifold $X_\tau = C^2(\mathbb{R}/\tau\mathbb{Z}, M)$. It is very unfortunate that the (PS) condition cannot be satisfied on $X_\tau$ though $\mathcal{L}_\tau$ is $C^2$ on it. So far one do not find a suitable space on which the functional $\mathcal{L}_\tau$ is not only $C^2$ but also satisfies the (PS) condition.

The common points of the two functionals above are: one hand on a Hilbert manifold they have smoothness lower than $C^2$, but satisfy the (PS) condition; on the other hand their critical point are contained in a densely and continuously imbedded Banach manifold on which the functional possesses at least $C^2$ smoothness, but does not satisfy the (PS) condition. To my knowledge there is no a suitable splitting lemma, which can be used to deal with the above functionals. These motivate us to look for a new splitting theorem.

With the regularity theory and prior estimation techniques of differential equations our theory can also be applied to some variational problems not satisfying our theorems (such as general Tonelli Lagrangian systems and geodesics on Finsler manifolds, see [32, Remarks 5.9,6.1] and the references cited therein) by modifying the original Euler-Lagrangian functions.

## 1.2 Notion and terminology

Since there often exists some small differences in references we state some necessary notions and terminologies for reader's conveniences. Let $E_1$ and $E_2$ be two real normed

linear spaces. Denote by $L(E_1,E_2)$ the space of the continuous linear operator from $E_1$ to $E_2$, and by $L(E_1)=L(E_1,E_1)$. A map $T$ from an open subset $U$ of $E_1$ to $E_2$ is called *directional differentiable* at $x\in U$ if for every $u\in E_1$ there exists an element of $E_2$, denoted by $DT(x,u)$, such that $\lim_{t\to 0}\frac{1}{t}\|T(x+tu)-T(x)-DT(x,u)\|=0$; $DT(x,u)$ is called the *directional derivative* of $T$ at $x$ in the direction $u$. If the map $U\times E_1\to E_2, (x,u)\mapsto DT(x,u)$ is continuous we say $T$ to be *continuously directional differentiable* on $U$. (This implies that $T$ is Gâteaux differentiable at every point of $U$ in the following sense). If there exists a $B\in L(E_1,E_2)$ such that $DT(x_0,u)=Bu\ \forall u\in E_1$, $T$ is called *Gâteaux differentiable* at $x_0\in U$, and $B$ is called the *Gâteaux derivative* of $T$ at $x_0$, denoted by $DT(x_0)$ (or $T'(x_0)$). By Definition 3.2.2 of [42], $T$ is called *strictly* G (Gâteaux) *differentiable* at $x_0\in U$ if for any $v\in E_1$,

$$\|T(x+tv)-T(x)-T'(x_0)(v)\|=o(|t|)\quad \text{as } x\to x_0 \text{ and } t\to 0;$$

if this convergence uniformly holds for $v$ in any compact subset we say $T$ to be *strictly* H (Hadamard) *differentiable* [3] at $x_0\in U$; moreover $T$ is called *strictly* [4] F (Fréchet) *differentiable* at $x_0\in U$ if

$$\|T(x)-T(y)-T'(x_0)(x-y)\|=o(\|x-y\|)\quad \text{as } x\to x_0 \text{ and } y\to x_0$$

(this implies that $T$ has Fréchet derivative $T'(x_0)$ at $x_0$). By [15, Prop.2.2.1] or [42, Prop.3.2.4(iii)], $T$ is strictly H-differentiable at $x_0\in U$ if and only if $T$ is locally Lipschitz continuous around $x_0$ and strictly G-differentiable at $x_0\in U$. Specially, the strict $F$-differentiability of $T$ at $x_0$ implies that $T$ is Lipschitz continuous in some neighborhood of $x_0$. By [42, Prop.3.4.2], the continuous F-differentiability of $T$ at $x_0$ implies that $T$ is strictly F-differentiable at $x_0$. If $T$ is $F$-differentiable in $U$, then $dT=T'$ is continuous at $x_0\in U$ (i.e. $T$ is continuously differentiable at $x_0$) if and only if $T$ is strictly $F$-differentiable at $x_0$, see Questions 3a) and 7a) at the end of [18, Chap.8, §6]. By Proposition B.1 the continuously directional differentiability of $T$ in $U$ implies the strict $H$-differentiability of $T$ in $U$ (and thus the locally Lipschitz continuality of $T$ in $U$).

## 1.3 Method and overview

The main methods to the splitting lemma in past references are the implicit function theorem method such as [23] and dynamical system one as in [11, Th. 5.1] and [37, Th.8.3]. Our method is different from theirs completely. Recently, Duc-Hung-Khai [19] gave a new proof to the Morse-Palais lemma based on elementary differential calculus. It seems that the parameterized versions of the new Morse lemma cannot be applied to the above

[3] This is called *strictly differentiable* in [15, page 30].

[4] It is also called strongly F-differentiable in some books, for instance. Question 7) at the end of [18, Chap.8, §6].

two typical functionals yet. After carefully analyzing the functionals we combine it with some techniques from [27, 43, 46] to successfully design a splitting lemma which is applicable to our above functionals. For completeness and reader's convenience we state the parameterized versions of Duc-Hung-Khai's Morse-Palais lemma in [19] and outline its proof in Appendix A. Some results on functional analysis are given in Appendix B.

In Section 2 we state our main results, which include a new splitting lemma, Theorem 2.1, and the corresponding shifting theorem, Corollary 2.6. We also obtain critical group characteristics for local minimum and critical points of mountain pass type under weaker conditions in Corollaries 2.7, 2.9, respectively. Corollary 2.5 and Theorem 2.10 study relations between critical groups of a functional and its restriction on a densely imbedded Banach space, which are very key for our current work [35]. A theorem of Poincaré-Hopf type, Theorem 2.12, is proved in Section 5. We also study the functor properties of our splitting lemma in Section 6, and estimate behavior of the functional $\mathcal{L}$ of Theorem 2.1 near $\theta$ in Section 7. As concluding remarks it is shown in Section 8 that the most results in Theorem 2.1 still hold true under weaker conditions.

These result have been used in [34] to generalize some previous results on computations of critical groups and some critical point theorems to weaker versions.

This paper consists of the sections 1,2 and the appendix of [33], which is not to be published elsewhere. The fourth section of [33] has been rewritten and extended into a separate paper. The author would like to express his deep gratitude to the anonymous referee for many valuable revision suggestions and for pointing out many misprints.

## 2 Statements of main results

Let $H$ be a Hilbert space with inner product $(\cdot,\cdot)_H$ and the induced norm $\|\cdot\|$, and let $X$ be a Banach space with norm $\|\cdot\|_X$, such that

**(S)** $X\subset H$ is dense in $H$ and the inclusion $X\hookrightarrow H$ is continuous, i.e. we may assume $\|x\|\le\|x\|_X$ $\forall x\in X$.

For an open neighborhood $V$ of the origin $\theta\in H$, $V\cap X$ is also an open neighborhood of $\theta$ in $X$, denoted by $V^X$ for clearness without special statements. Suppose that a functional $\mathcal{L}:V\to\mathbb{R}$ satisfies the following conditions:

**(F1)** $\mathcal{L}$ is continuously directional differentiable (and thus $C^{1-0}$) on $V$.

**(F2)** There exists a continuously directional differentiable (and thus $C^{1-0}$) map $A:V^X\to X$, which is strictly Fréchet differentiable at $\theta$, such that

$$D\mathcal{L}(x)(u)=(A(x),u)_H\quad\forall x\in V^X\text{ and }u\in X.$$

(This actually implies that $\mathcal{L}|_{V^X}\in C^1(V^X,\mathbb{R})$.)

**(F3)** There exists a map $B$ from $V^X$ to the space $L_s(H)$ of bounded self-adjoint linear operators of $H$ such that

$$(DA(x)(u), v)_H = (B(x)u, v)_H \ \forall x \in V^X \text{ and } u, v \in X.$$

(This and (F1)-(F2) imply: (a) $A$ is Gâtuax differentiable and $DA(x) = B(x)|_X$ for all $x \in V^X$, (b) $B(x)(X) \subset X \ \forall x \in V^X$, (c) $d(\mathcal{L}|_{V^X})$ is strictly Frechét differentiable at $\theta \in V^X$, and $d^2(\mathcal{L}|_{V^X})(\theta)(u, v) = (B(\theta)u, v)_H$ for any $u, v \in X$.)

**(C1)** The origin $\theta \in X$ is a critical point of $\mathcal{L}|_{V^X}$ (and thus $\mathcal{L}$), $0$ is either not in the spectrum $\sigma(B(\theta))$ or is an isolated point of $\sigma(B(\theta))$. [5]

**(C2)** If $u \in H$ such that $B(\theta)(u) = v$ for some $v \in X$, then $u \in X$.

**(D)** The map $B : V^X \to L_s(H)$ has a decomposition [6]

$$B(x) = P(x) + Q(x) \quad \forall x \in V^X,$$

where $P(x) : H \to H$ is a positive definitive linear operator and $Q(x) : H \to H$ is a compact linear operator with the following properties:

- **(D1)** All eigenfunctions of the operator $B(\theta)$ that correspond to negative eigenvalues belong to $X$;
- **(D2)** For any sequence $\{x_k\} \subset V \cap X$ with $\|x_k\| \to 0$ it holds that $\|P(x_k)u - P(\theta)u\| \to 0$ for any $u \in H$;
- **(D3)** The map $Q : V \cap X \to L(H)$ is continuous at $\theta$ with respect to the topology induced from $H$ on $V \cap X$;
- **(D4)** For any sequence $\{x_n\} \subset V \cap X$ with $\|x_n\| \to 0$ (as $n \to \infty$), there exist constants $C_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$(P(x_n)u, u)_H \geq C_0\|u\|^2 \quad \forall u \in H, \ \forall n \geq n_0.$$

Sometimes we need to replace the condition (D4) by the following slightly stronger

**(D4*)** There exist positive constants $\eta_0 > 0$ and $C_0' > 0$ such that

$$(P(x)u, u) \geq C_0'\|u\|^2 \quad \forall u \in H, \ \forall x \in B_H(\theta, \eta_0) \cap X.$$

[5] The claim in the latter sentence is actually implied in the following condition (D) by Proposition B.2. In order to state some results without the condition (D) we still list it.

[6] Actually, this and (D4) imply the claim in the second sentence in (C1) by Proposition B.2.

Here is a way looking for the map $B$. Suppose that $\mathcal{L}|_{V^X}$ is twice Gâteaux differentiable at every point $x\in V^X$, i.e. for any $u_1,u_2\in X$ the limit

$$D\mathcal{L}|_{V^X}(x;u_1,u_2)=\lim_{t_2\to 0}\lim_{t_1\to 0}\frac{1}{t_1t_2}\triangle^2_{t_1u_1,t_2u_2}\mathcal{L}(x)$$

exists and is linear continuous with respect to $u_i$, $i=1,2$, where

$$\triangle^2_{t_1u_1,t_2u_2}\mathcal{L}(x)=\mathcal{L}(x+t_1u_1+t_2u_2)-\mathcal{L}(x+t_1u_1)-\mathcal{L}(x+t_2u_2)-\mathcal{L}(x).$$

By (F2) the map $A:V^X\to X$ is Gâteaux differentiable and

$$D\mathcal{L}|_{V^X}(x;u_1,u_2)=(A'(x)u_2,u_1)_H\quad\forall x\in V^X,\ u_1,u_2\in X.$$

If $(u_1,u_2)\mapsto D\mathcal{L}|_{V^X}(x;u_1,u_2)$ is symmetric then $A'(x)\in L(X)$ is self-adjoint with respect to the inner $(\cdot,\cdot)_H$. By Question 17) at the end of [18, Chap.11, §5], $A'(x)$ can be extended into an element $\hat{B}(x)\in L_s(H)$ with the following properties: (a) $\|\hat{B}(x)\|_{L(H)}\le\rho_X(A'(x))\le\|A'(x)\|_{L(X)}$ and $\sigma(\hat{B}(x))\subset\sigma(A'(x))$, (b) if $A'(x)$ is compact in $(X,\|\cdot\|_X)$ so is $\hat{B}(x)$ in $(H,\|\cdot\|)$. In the case, if $B$ is a map satisfying the conditions (F3), (C1)-(C2) and (D), it holds that $B(x)=\hat{B}(x)\ \forall x\in V^X$.

By the assumption (D) each $B(x)$ is Fredholm. In particular, $H^0:=\mathrm{Ker}(B(\theta))$ is finitely dimensional. Let $H^\pm:=(H^0)^\perp$ be the range of $B(\theta)$. There exists an orthogonal decomposition $H=H^0\oplus H^\pm=H^0\oplus H^-\oplus H^+$, where $H^-$ and $H^+$ are subspaces invariant under $B(\theta)$ such that $B(\theta)|_{H^+}$ is positive definite and $B(\theta)|_{H^+}$ is negative definite. Clearly, we have also

$$\left.\begin{array}{l}(B(\theta)u,v)_H=0\ \forall u\in H^+\oplus H^-,\ v\in H^0,\\ (B(\theta)u,v)_H=0\ \forall u\in H^-\oplus H^0,\ v\in H^+,\\ (B(\theta)u,v)_H=0\ \forall u\in H^+\oplus H^0,\ v\in H^-.\end{array}\right\}\tag{2.1}$$

By the condition (C1) there exists a small $a_0>0$ such that $[-2a_0,2a_0]\cap\sigma(B(\theta))$ at most contains a point $0$. Hence

$$\left.\begin{array}{l}(B(\theta)u,u)_H\ge 2a_0\|u\|^2\quad\forall u\in H^+,\\ (B(\theta)u,u)_H\le -2a_0\|u\|^2\quad\forall u\in H^-.\end{array}\right\}\tag{2.2}$$

The conditions (C2) and (D) imply that both $H^0$ and $H^-$ are finitely dimensional subspaces contained in $X$ by Proposition B.2. Denote by $P^*$ the orthogonal projections onto $H^*$, $*=+,-,0$, and by $X^*=X\cap H^*=P^*(X)$, $*=+,-$. Then $X^+$ is dense in $H^+$, and $(I-P^0)|_X=(P^++P^-)|_X:(X,\|\cdot\|_X)\to(X^\pm,\|\cdot\|)$ is also continuous because all norms are equivalent on a linear space of finite dimension, where $X^\pm:=X\cap(I-P^0)(H)=X\cap H^\pm=X^-+P^+(X)=X^-+H^+\cap X$. These give the following topological direct sum decomposition:

$$X=H^0\oplus X^\pm=H^0\oplus X^+\oplus X^-.$$

Let $\nu = \dim H^0$ and $\mu = \dim H^-$. We call them the *nullity* and the *Morse index* of critical point $\theta$ of $\mathcal{L}$, respectively. In particular, the critical point $\theta$ is said to be *nondegenerate* if $\nu = 0$. Since the norms $\|\cdot\|$ and $\|\cdot\|_X$ are equivalent on the finite dimension space $H^0$ we shall not point out the norm used without occurring of confusions. In this paper, for a normed vector space $(E, \|\cdot\|)$ and $\delta > 0$ let $B_E(\theta, \delta) = \{x \in E \,:\, \|x\| = \|x - \theta\| < \delta\}$ and $\bar{B}_E(\theta, \delta) = \{x \in E \,:\, \|x\| \le \delta\}$. Moreover, we always use $\theta$ to denote the origins of all linear spaces without occurring of confusions.

**Theorem 2.1.** *Under the above assumptions* (S), (F1)-(F3) *and* (C1)-(C2), (D), *if* $\nu > 0$ *there exist a positive* $\epsilon \in \mathbb{R}$, *a (unique) Lipschitz continuous map* $h : B_{H^0}(\theta, \epsilon) = B_H(\theta, \epsilon) \cap H^0 \to X^{\pm}$ *satisfying* $h(\theta) = \theta$ *and*

$$(I - P^0)A(z + h(z)) = 0 \quad \forall z \in B_{H^0}(\theta, \epsilon), \tag{2.3}$$

*an open neighborhood* $W$ *of* $\theta$ *in* $H$ *and an origin-preserving homeomorphism*

$$\Phi : B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon)) \to W \tag{2.4}$$

*of form* $\Phi(z, u^+ + u^-) = z + h(z) + \phi_z(u^+ + u^-)$ *with* $\phi_z(u^+ + u^-) \in H^{\pm}$ *such that*

$$\mathcal{L} \circ \Phi(z, u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 + \mathcal{L}(z + h(z)) \tag{2.5}$$

*for all* $(z, u^+ + u^-) \in B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon))$, *and that*

$$\Phi\left(B_{H^0}(\theta, \epsilon) \times \left(B_{H^+}(\theta, \epsilon) \cap X + B_{H^-}(\theta, \epsilon)\right)\right) \subset X. \tag{2.6}$$

*Moreover, the homeomorphism* $\Phi$ *has also properties:*

- **(a)** *For each* $z \in B_{H^0}(\theta, \epsilon)$, $\Phi(z, \theta) = z + h(z)$, $\phi_z(u^+ + u^-) \in H^-$ *if and only if* $u^+ = \theta$;
- **(b)** *The restriction of* $\Phi$ *to* $B_{H^0}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon)$ *is a homeomorphism from* $B_{H^0}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon) \subset X \times X$ *onto* $\Phi(B_{H^0}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon)) \subset X$ *even if the topologies on these two sets are chosen as the induced one by* $X$.

*The map* $h$ *and the function* $B_{H^0}(\theta, \epsilon) \ni z \mapsto \mathcal{L}^\circ(z) := \mathcal{L}(z + h(z))$ [7] *also satisfy:*

- **(i)** *The map* $h$ *is strictly Fréchet differentiable at* $\theta \in H^0$ *and*

$$h'(\theta)z = -[(I - P^0)A'(\theta)|_{X^{\pm}}]^{-1}(I - P^0)A'(\theta)z \quad \forall z \in H^0;$$

- **(ii)** $\mathcal{L}^\circ$ *is* $C^{2-0}$,

$$d\mathcal{L}^\circ(z_0)(z) = (A(z_0 + h(z_0)), z)_H \quad \forall z_0 \in B_{H^0}(\theta, \epsilon),\ z \in H^0,$$

  *and* $d\mathcal{L}^\circ$ *is strictly F-differentiable at* $\theta \in H^0$ *and* $d^2\mathcal{L}^\circ(\theta) = 0$;

[7] If $A$ is $C^1$ then maps $h$ and $\mathcal{L}^\circ$ have higher smoothness too, see Remark 3.2.

**(iii)** *If $\theta$ is an isolated critical point of $\mathcal{L}|_{V^X}$, then $\theta$ is also an isolated critical point of $\mathcal{L}^\circ$.*

If the strictly Fréchet differentiability at $\theta$ of the map $A : V^X \to X$ in (F2) is replaced by weaker conditions we shall show in Section 8 that the most results in Theorem 2.1 still hold true.

Under the conditions (L1)-(L3) it was proved in [32] that the functional $\mathcal{L}_\tau$ in (1.2) satisfies the assumptions of Theorem 2.1 near a critical point of it. In fact, a special version of Theorem 2.1 was used there. As stated in [43, §5.2] the arguments of [43, Chap.3] showed that the functional $F$ in (1.1) satisfies the assumptions of Theorem 2.1 near a critical point of it too. Our frame conditions in Theorem 2.1 seem strange and complex. But they come from abstract and analysis for the studies in [43]. Of course, the theory of this paper can be used to improve one of [43]. This work is in progress.

**Remark 2.2.** (i) Note that our proof only use the Banach fixed point theorem or the implicit function theorem in the case $H^0 \neq \{0\}$. If $H^0 = \{0\}$, we do not require the completeness of $(X, \|\cdot\|_X)$, that is, the condition (S) can be replaced by the following

**(S')** $(X, \|\cdot\|_X)$ is a normed vector space, $X \subset H$ is dense in $H$ and the inclusion $X \hookrightarrow H$ is continuous, i.e. we may assume $\|x\| \le \|x\|_X$ $\forall x \in X$;

And the conclusions of Theorem 2.1 become: There exist a positive $\epsilon \in \mathbb{R}$, an open neighborhood $W$ of $\theta$ in $H$ and an origin-preserving homeomorphism, $\phi : B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon) \to W$, such that

$$\mathcal{L} \circ \phi(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 \tag{2.7}$$

for all $(u^+, u^-) \in B_{H^+}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon)$, and that

$$\phi\left((B_{H^+}(\theta, \epsilon) \cap X) + B_{H^-}(\theta, \epsilon)\right) \subset X.$$

Moreover, $\phi(u^+ + u^-) \in H^-$ if and only if $u^+ = \theta$, and the restriction of $\phi$ to $B_{H^-}(\theta, \epsilon)$ is a homeomorphism from $B_{H^-}(\theta, \epsilon) \subset X$ onto $\phi(B_{H^-}(\theta, \epsilon)) \subset X$ even if the topologies on $B_{H^-}(\theta, \epsilon) \subset X$ and $\phi(B_{H^-}(\theta, \epsilon)) \subset X$ are chosen as the induced ones by $X$.
(ii) Suppose that $\mathcal{L}$ is only defined on $V \cap X$ and that the condition (F1) can be replaced by the following

**(F1')** $\mathcal{L}$ is continuously directional differentiable (and so $C^{1-0}$) on $V \cap X$ with respect to the topology of $H$.

Then the origin-preserving homeomorphism in (2.4) should be changed into

$$\Phi : B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, \epsilon) \cap X + B_{H^-}(\theta, \epsilon)) \to W \cap X \tag{2.8}$$

(with respect to the topology of $H$), which satisfies (2.5) for all $(z, u^+, u^-) \in B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, \epsilon) \cap X + B_{H^-}(\theta, \epsilon))$.

**Remark 2.3.** Since Lemmas 3.3, 3.4 are only used in the proof of Lemma 3.5. Carefully checking the proof of the latter one easily see that the condition (D) can be replaced by the following

**(D')** There exist a small neighborhood $U \subset V$ of $\theta$ in $H$, a positive number $c_0$ and a function $\omega : U \cap X \to [0, \infty)$ with property $\omega(x) \to 0$ as $x \in U \cap X$ and $\|x\| \to 0$, to satisfy

- ($\mathrm{D}'_1$) The kernel $H^0$ and negative definite subspace $H^-$ of $B(\theta)$ are finitely dimensional subspaces contained in $X$; [8]
- ($\mathrm{D}'_2$) $(B(x)v, v)_H \ge c_0\|v\|^2\ \forall v \in H^+$;
- ($\mathrm{D}'_3$) $|(B(x)u, v)_H - (B(\theta)u, v)_H| \le \omega(x)\|u\| \cdot \|v\| \quad \forall u \in H, v \in H^- \oplus H^0$;
- ($\mathrm{D}'_4$) $(B(x)u, u)_H \le -c_0\|u\|^2\ \forall u \in H^-$.

**Remark 2.4.** When $(X, \|\cdot\|_X) = (H, \|\cdot\|)$ the conditions (F1)-(F3) are reduced to:

**(F)** $\mathcal{L}$ is $C^1$, $\nabla\mathcal{L}$ is continuously directional differentiable (and so Gâteaux differentiable) in $V$ and strictly Fréchet differentiable at $\theta \in H$, and $B(x) := D(\nabla\mathcal{L})(x) \in L_s(H)$ for any $x \in V$.

Clearly, this holds if $\mathcal{L} \in C^2(V, \mathbb{R})$. In fact, the condition (C1) for $B(\theta) = d^2\mathcal{L}(\theta)$ also imply the condition (D) in the case $\dim H^0 \oplus H^- < \infty$. In order to see this we can write $B(x) = P(x) + Q(x)$, where $P(x) = P^+B(x) - P^-B(x) + P^0$ and $Q(x) = 2P^-B(x) + P^0 + P^0B(x)$. The latter is finite rank and therefore compact. The continuity of the map $B : V \to L_s(H)$ implies that both maps $P$ and $Q$ are continuous, and that there exists a $\delta > 0$ such that

$$\|B(x) - B(\theta)\|_{L(H)} < \min\{a_0, 1\}/4 \quad \forall x \in B_H(\theta, \delta).$$

Note that $(P(\theta)u, u)_H \ge \min\{a_0, 1\}\|u\|^2\ \forall u \in H$ and that

$$|(P(x)u, u)_H - (P(\theta)u, u)_H| \le 2\|B(x) - B(\theta)\|_{L(H)} \cdot \|u\|^2 \quad \forall u \in H.$$

We get

$$(P(x)u, u)_H \ge \frac{\min\{a_0, 1\}}{2}\|u\|^2\ \forall u \in H.$$

These show that the condition (D) is satisfied. Hence Theorem 2.1 is a generalization of [24, Th.3] and [37, Th.8.3], [31, Th.2.2], and [11, Th.5.1. p.44] in the case $\dim H^0 \oplus H^- < \infty$ (a condition naturally satisfied in applications). Since the strictly Fréchet differentiability of $\nabla\mathcal{L}$ at $\theta \in H$ implies that $\nabla\mathcal{L}$ is $C^{1-0}$ near $\theta$, we cannot guarantee that Theorem 2.1 include [26, Cor.3]. (*Note*: By [16, Th.4.5] the assumptions in [38, Th.1.2] is actually the same as that of [26, Cor.3], but the author cannot verify the equalities $h_2 \circ h_3 = id = h_3 \circ h_2$ below (2.19) of [38].)

[8] It seems to be sufficient for us to assume only that $H^0 \subset X$ and is closed in $X$.

For an open neighborhood $W$ of $\theta$ in $H$, we write $W^X = W \cap X$ as an open neighborhood of $\theta$ in $X$. Note that $(\mathcal{L}|_{V^X})_0 \cap (W \cap X) = (\mathcal{L}|_{V^X})_0 \cap W = \mathcal{L}_0 \cap W^X$.

**Corollary 2.5.** *For any Abel group* $\mathbf{K}$ *and an open neighborhood* $W$ *of* $\theta$ *in* $H$*, the inclusion*

$$I^{xw} : (\mathcal{L}_0 \cap W^X, \mathcal{L}_0 \cap W^X \setminus \{\theta\}) \hookrightarrow (\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\})$$

*induce surjective homomorphisms*

$$H_*(\mathcal{L}_0 \cap W^X, \mathcal{L}_0 \cap W^X \setminus \{\theta\}; \mathbf{K}) \to H_*(\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}; \mathbf{K}).$$

Hereafter $H_q(A, B; \mathbf{K})$ denotes the $q$th relative singular homology group of a pair $(A, B)$ of topological spaces with coefficients in $\mathbf{K}$.

One of important applications of the splitting lemma is to compute critical groups of critical points. Recall that for $q \in \mathbb{N} \cup \{0\}$ *the $q$th critical group* (with coefficients in $\mathbf{K}$) of a real continuous functional $f$ on a metric space $\mathcal{M}$ at a point $x \in \mathcal{M}$ is defined by

$$C_q(f, x; \mathbf{K}) = H_q(f_c \cap U, f_c \cap U \setminus \{x\}; \mathbf{K}),$$

where $c = f(x)$ and $U$ is a neighborhood of $x$ in $\mathcal{M}$. The definition of the critical groups are independent of the special choice of $U$ because of the excision property of the singular homology. If $\mathcal{M}$ is a Banach space and $f$ is $C^1$ then the $q$th critical group of an isolated critical point $x$ may equivalently be defined as

$$C_q(f, x; \mathbf{K}) = H_q((f_c^\circ \cup \{x\}) \cap U, f_c^\circ \cap U; \mathbf{K}),$$

where $c = f(x)$, $f_c^\circ = \{f < c\}$ and $U$ is as above. (See [17, Prop.3.7]).

If the critical point $\theta$ of $\mathcal{L}$ is isolated, then it is also an isolated critical point of $\mathcal{L}|_{V^X}$. By Theorem 2.1 $\theta \in H^0$ is an isolated critical point of $\mathcal{L}^\circ$. Since $\mathcal{L}^\circ$ is also $C^{2-0}$ and $\dim H^0 < \infty$ we can construct a $C^{2-0}$ function on $H^0$ that satisfies the (PS) condition and is equal to $\mathcal{L}^\circ$ near $\theta$. With the same proof method as in [37, Th.8.4] or [12, Th.5.1.17] we can use Theorem A.1 to derive:

**Corollary 2.6** (Shifting)**.** *Under the assumptions of Theorem 2.1, if $\theta$ is an isolated critical point of $\mathcal{L}$, for any Abel group $\mathbf{K}$ it holds that*

$$C_q(\mathcal{L}, \theta; \mathbf{K}) \cong C_{q-\mu}(\mathcal{L}^\circ, \theta; \mathbf{K}) \quad \forall q = 0, 1, \cdots,$$

*where $\mathcal{L}^\circ(z) = \mathcal{L}(h(z) + z)$.* (*Consequently, $C_q(\mathcal{L}, \theta; \mathbf{K}) = 0$ for $q \notin [\mu, \mu + \nu]$, and $C_q(\mathcal{L}, \theta; \mathbf{K})$ is isomorphic to a finite direct sum $r_1\mathbf{K} \oplus \cdots \oplus r_s\mathbf{K}$ for each $q \in [\mu, \mu + \nu]$, where each $r_j \in \{0, 1\}$, see Proposition 4.5.*)

Corresponding with Proposition 3.2 of [3], but no requirement for the (PS) condition, we have

**Corollary 2.7.** *Under the assumptions of Theorem 2.1, if $\theta$ is an isolated critical point of $\mathcal{L}$, the following are equivalent.*

**(i)** *$\theta$ is a local minimum;*

**(ii)** $C_q(\mathcal{L}, \theta; \mathbf{K}) \cong \delta_{q0}\mathbf{K} \quad \forall q \in \mathbb{Z}$*;*

**(iii)** $C_0(\mathcal{L}, \theta; \mathbf{K}) \neq 0$.

Actually our proof shows that (iii) implies $\theta$ to be a strict minimum.

Since $d^2\mathcal{L}|_{V^X}(\theta)(u, v) = (B(\theta)u, v)_H \ \forall u, v \in X$ we arrive at $H^0 = \{\theta\} = H^-$ provided that $d^2(\mathcal{L}|_{V^X})(\theta)(u, u) > 0$ for any $u \in X \setminus \{\theta\}$. From Theorem 2.1 or Step 3 in the proof of Lemma 3.5 we easily derive a similar conclusion of Tromba's main result Theorem 1.3 in [45] without requirement for completeness of $(X, \|\cdot\|_X)$.

**Corollary 2.8.** *Under the assumptions of Theorem 2.1, but no requirement for completeness of $(X, \|\cdot\|_X)$, i.e., the condition* (S) *is replaced by* (S′)*, suppose also that $d^2(\mathcal{L}|_{V^X})(\theta)(u, u) > 0$ for any $u \in X \setminus \{\theta\}$. Then $\theta$ is a strict minimum for $\mathcal{L}$ and thus $\mathcal{L}|_{V^X}$.*

According to Hofer [24] the critical point $\theta$ is called *mountain pass type* if for any small neighborhood $\mathcal{O}$ of $\theta$ in $H$ the set $\{x \in \mathcal{O} \,|\, \mathcal{L}(x) < 0\}$ is nonempty and not path-connected.

**Corollary 2.9.** *Under the assumptions of Theorem 2.1 (and hence without the* (PS) *condition), let $\theta$ be an isolated critical point of $\mathcal{L}$ with Morse index $\mu$ and nullity $\nu$.*

**(i)** *If $C_1(\mathcal{L}, \theta; \mathbf{K}) \neq 0$ and $\nu = \dim \mathrm{Ker}(B(\theta)) = 1$ then*

$$C_q(\mathcal{L}, \theta; \mathbf{K}) \cong \delta_{q1}\mathbf{K} \ \forall q \in \mathbb{Z};$$

**(ii)** *If $\nu = \dim \mathrm{Ker}(B(\theta)) = 1$ in the case $\mu = \dim H^- = 0$, then $\theta$ is mountain pass type if and only if $C_q(\mathcal{L}, \theta; \mathbf{K}) \cong \delta_{q1}\mathbf{K} \ \forall q \in \mathbb{Z}$;*

**(iii)** *If $C_\mu(\mathcal{L}, \theta; \mathbf{K}) \neq 0$, then $C_q(\mathcal{L}, \theta; \mathbf{K}) \cong \delta_{q\mu}\mathbf{K} \ \forall q \in \mathbb{Z}$.*

The proofs of (i) and (ii) are the same as those of [11, Th.II.1.6] and [3, Prop.3.3], respectively, with some slight replacements by Theorem 2.1. (iii) corresponds to Proposition 2.4 in [2] and can be proved similarly. (Note that Theorem 4.6 in [11, page. 43] does not need the (PS) condition in finite dimension space.) Since (F1) implies that $\mathcal{L} : V \to \mathbb{R}$ is

Gâteaux differentiable, if $V = X$ and $D\mathcal{L} : X \to X^*$ is continuous from the norm topology of $X$ to the weak*-topology of $X^*$ one may use a generalized version of mountain pass lemma in [22] to yield a critical point of mountain pass type provided that $\mathcal{L}$ also satisfies the condition (C) (weaker than (PS)).

If the critical point $\theta$ of $\mathcal{L}$ is isolated, Corollary 2.5 yields surjective homomorphisms from critical groups $C_*(\mathcal{L}|_{V^X}, \theta; \mathbf{K})$ to $C_*(\mathcal{L}, \theta; \mathbf{K})$, which are also isomorphisms provided that $\mathbf{K}$ is a field and both groups are finite dimension vector spaces over $\mathbf{K}$ of same dimension. When $\mathcal{L} \in C^2(V, \mathbb{R})$ and $A \in C^1(V^X, X)$ it follows from [27, Cor.2.8] that $C_*(\mathcal{L}|_{V^X}, \theta; \mathbf{K}) \cong C_*(\mathcal{L}, \theta; \mathbf{K})$ for any Abel group $\mathbf{K}$. The following theorem generalizes and refines this.

**Theorem 2.10.** *Under the assumptions of Theorem 2.1, let $\theta \in H$ be an isolated critical point of $\mathcal{L}$ and let $(Y, \|\cdot\|_Y)$ be another Banach space such that $X \subset Y \subset H$ and that $(X, \|\cdot\|_X)$ is a densely embedded Banach space in $(Y, \|\cdot\|_Y)$ (and hence $(Y, \|\cdot\|_Y)$ is a densely embedded Banach space in $(H, \|\cdot\|)$ due to* (S)*. We may assume that $\|y\| \le \|y\|_Y$ $\forall y \in Y$ and $\|x\|_Y \le \|x\|_X$ $\forall x \in X$). For an open neighborhood $V$ of the origin $\theta \in H$, write $V^X = V \cap X$ (resp. $V^Y = V \cap Y$) as an open subset of $X$ (resp. $Y$) as before. Assume also that*

- **(i)** $\mathcal{L}|_{V^Y} \in C^2(V^Y, \mathbb{R})$.
- **(ii)** *The map $A$ in* (F2) *belongs to $C^1(V^X, X)$.* [9]
- **(iii)** *The map $B$ in* (F2) *can be extended into a continuous map $B : V^Y \to L_s(H)$ satisfying*

$$d^2(\mathcal{L}|_{V^Y})(y)(u, v) = (B(y)u, v)_H \quad \forall y \in V^Y \text{ and } u, v \in Y.$$

*Then for any open neighborhood $W$ of $\theta$ in $V$ and a field $\mathbb{F}$ the inclusions*

$$I^{xw} : \left(\mathcal{L}_0 \cap W^X, \mathcal{L}_0 \cap W^X \setminus \{\theta\}\right) \to \left(\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}\right),$$
$$I^{yw} : \left(\mathcal{L}_0 \cap W^Y, \mathcal{L}_0 \cap W^Y \setminus \{\theta\}\right) \to \left(\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}\right)$$

*induce isomorphisms*

$$I^{xw}_* : H_*\left(\mathcal{L}_0 \cap W^X, \mathcal{L}_0 \cap W^X \setminus \{\theta\}; \mathbb{F}\right) \to H_*\left(\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}; \mathbb{F}\right),$$
$$I^{yw}_* : H_*\left(\mathcal{L}_0 \cap W^Y, \mathcal{L}_0 \cap W^Y \setminus \{\theta\}; \mathbb{F}\right) \to H_*\left(\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}; \mathbb{F}\right).$$

*Consequently, $C_*(\mathcal{L}|_{V^X}, \theta; \mathbb{F}) \cong C_*(\mathcal{L}|_{V^Y}, \theta; \mathbb{F}) \cong C_*(\mathcal{L}, \theta; \mathbb{F})$.*

The first isomorphism in the final claims is due to Jiang [27], see Corollary 4.4. Taking $Y = X$ we get

[9] This and (i) imply $\mathcal{L}|_{V^X} \in C^2(V^X, \mathbb{R})$.

**Corollary 2.11.** *Under the assumptions of Theorem 2.1, also assume:* (i) $\theta$ *is an isolated critical point of* $\mathcal{L}$, (ii) $\mathcal{L}|_{V^X} \in C^2(V^X, \mathbb{R})$, (iii) *the map* $A$ *in* (F2) *belongs to* $C^1(V^X, X)$, (iv) *the map* $B$ *in* (F3) *is continuous, Then for any open neighborhood* $W$ *of* $\theta$ *in* $V$ *and a field* $\mathbb{F}$ *the inclusion*

$$I^{xw}: \left(\mathcal{L}_0 \cap W^X, \mathcal{L}_0 \cap W^X \setminus \{\theta\}\right) \to \left(\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}\right),$$

*induces isomorphisms between their relative homology groups with coefficients in* $\mathbb{F}$. *Specially,* $C_*(\mathcal{L}|_{V^X}, \theta; \mathbb{F}) \cong C_*(\mathcal{L}, \theta; \mathbb{F})$.

If $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary $\partial\Omega$, and $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies the condition: $|f_t'(x,t)| \le C(1 + |t|^\alpha)$ for some constants $C > 0$ and $\alpha \le \frac{n+2}{n-2}$ (if $n > 2$), then for an isolated critical point $u_0$ of the functional

$$J(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - F(x,u)\Big) dx$$

(where $F$ is the primitive of $f$ with respect to $u$) on $H = H_0^1(\Omega)$ it follows from Corollary 2.11 that $C_*(J, u_0; \mathbb{K}) \cong C_*(J|_X, u_0; \mathbb{K})$ provided that $u_0 \in X = C_0^1(X)$ is also an isolated critical point of $J|_X$. This result was obtained by Chang [13] under the assumption that $J$ satisfies the $(\mathrm{PS})_c$ condition. Brézis and Nirenberg [8] firstly proved it as $u_0$ is a minimizer.

Theorem 2.1 and Corollary 2.6 cannot be applied to the geodesic problems on Finsler geometry directly. But as outlined in Remark 5.9 of [32] we may develop an method of infinite dimensional Morse theory for geodesics on Finsler manifolds based on them in [35], that is, giving the shifting theorem of critical groups of the energy functional of a Finsler manifold at a nonconstant critical orbit and relations of critical groups under iterations. In particular, Corollary 2.5 is a key for us to realize the second goal.

Finally we give a theorem of Poincaré-Hopf type. By the condition (F1) the functional $\mathcal{L}: V \to \mathbb{R}$ is Gâteaux differentiable. Its gradient $\nabla\mathcal{L}$ is equal to $A$ on $V \cap X$ by the condition (F2). Furthermore, under the assumptions (F3) and (D) we can prove that for a small $\epsilon > 0$ the restriction of $\nabla\mathcal{L}$ to $B_H(\theta, 2\epsilon)$ has a unique zero $\theta$ and is a demicontinuous map of class $(S)_+$. According to [9] and [44] we have a degree $\deg_{\mathrm{BS}}(\nabla\mathcal{L}, B_H(\theta,\epsilon), \theta)$. Under the conditions (C1) and (C2), $A'(\theta): X \to X$ is a bounded linear Fredholm operator of index zero, see the first paragraph in Step 1 of proof of Lemma 3.1. If the map $A$ in (F2) is $C^1$, then $A$ is a Fredholm map of index zero near $\theta \in X$ and thus for sufficiently small $\epsilon > 0$ there exists a degree $\deg_{\mathrm{FPR}}(A, B_X(\theta,\epsilon), \theta)$ or $\deg_{\mathrm{BF}}(A, B_X(\theta,\epsilon), \theta)$ according to [21, 39] or [5, 6].

**Theorem 2.12.** *Under the assumptions of Theorem 2.1, one has:*

**(i)** *If the map $A$ in the condition $(\mathrm{F2})$ is $C^1$ near $\theta\in X$, then for small $\epsilon>0$*

$$\begin{aligned}\deg_{\mathrm{FPR}}(A,B_X(\theta,\epsilon),\theta)&=\deg_{\mathrm{BF}}(A,B_X(\theta,\epsilon),\theta)\\&=(-1)^\mu\deg(\nabla\mathcal{L}^\circ,B_X(\theta,\epsilon)\cap H^0,\theta)\\&=\sum_{q=0}^\infty(-1)^q\mathrm{rank}C_q(\mathcal{L},\theta;\mathbf{K})\end{aligned}$$

*provided a suitable orientation for $A$.*

**(ii)** *If $\theta$ is also an isolated critical point of $\mathcal{L}$, and the condition $(\mathrm{D4}^*)$ holds true, then for a small $\epsilon>0$,*

$$\begin{aligned}\deg_{\mathrm{BS}}(\nabla\mathcal{L},B_H(\theta,\epsilon),\theta)&=\sum_{q=0}^\infty(-1)^q\mathrm{rank}C_q(\mathcal{L},\theta;\mathbf{K})\\&=(-1)^\mu\sum_{q=0}^\infty(-1)^q\mathrm{rank}C_q(\mathcal{L}^\circ,\theta;\mathbf{K})\\&=(-1)^\mu\deg(\nabla\mathcal{L}^\circ,B_X(\theta,\epsilon)\cap H^0,\theta).\end{aligned}$$

*Here $\deg$ is the classical Brouwer degree.*

The first equality in (ii) of Theorem 2.12 is a direct consequence of [14, Th.1.2] once we prove that the map $\nabla\mathcal{L}$ is a demicontinuous map of class $(S)_+$ near $\theta\in H$.

Using Theorem 2.1 we also gave a handle body theorem under the our weaker framework in Theorem 2.8 of [34].

# 3 Proof of Theorem 2.1

We shall complete the proof of Theorem 2.1 by a series of lemmas.

**Lemma 3.1.** *Under the above assumption $(\mathrm{S})$, for an open neighborhood $V$ of $\theta\in H$ let $\mathcal{L}|_{V\cap X}:V\cap X\to\mathbb{R}$ be continuous and continuously directional differentiable* [10] *(with respect to the induced topology on $V\cap H$ from $H$). Let $B(\theta)\in L_s(H)$ satisfy the conditions* (C1) *and* (C2)*. Suppose that a map $A:V^X\to X$ is strictly F-differentiable at $\theta$ and satisfies $A'(\theta)=B(\theta)|_X$ and*

$$D\mathcal{L}(x)(u)=(A(x),u)_H\quad\forall x\in V\cap X\text{ and }u\in X.$$

*Then there exist a positive $r_0\in\mathbb{R}$, a unique map $h:B_{H^0}(\theta,r_0)\to X^\pm$ such that*

**(i)** *$h(\theta)=\theta$ and $(I-P^0)A(z+h(z))=\theta$ for all $z\in B_{H^0}(\theta,r_0)$;*

[10]The former can be derived from the latter with mean value theorem [42, Prop.3.3.3].

**(ii)** *$h$ is also Lipschitz continuous, strictly F-differentiable at $\theta \in H^0$ and $h'(\theta)z = \theta$ for any $z \in H^0$.*

*Moreover, the function $\mathcal{L}^\circ(z) = \mathcal{L}(z + h(z))$ is $C^{2-0}$,*

$$d\mathcal{L}^\circ(z_0)(z) = (A(z_0 + h(z_0)), z)_H \quad \forall z_0 \in B_{H^0}(\theta, r_0),\ z \in H^0,$$

*and $d\mathcal{L}^\circ$ is strictly F-differentiable at $\theta \in H^0$ and $d^2\mathcal{L}^\circ(\theta) = 0$. (Clearly, if $\theta$ is an isolated critical point of $\mathcal{L}|_{V^X}$ (thus an isolated zero of $A$) then $\theta$ is also an isolated critical point of $\mathcal{L}^\circ$.)*

*Proof.* The proof method seems to be standard. For completeness and the reader's conveniences we give its detailed proof in two steps.

**Step 1**. Since $B(\theta) \in L_s(H)$ and $A'(\theta) = B(\theta)|_X$ (so $B(\theta)(X) \subset X$), using (C1)-(C2) it was proved in [27] that $B(\theta)(X^\pm) \subset X^\pm$ and $B(\theta)|_{X^\pm} : X^\pm \to X^\pm$ is an isomorphism. (*Note*: It is where the assumption (C1) is used to prove that the range $R(B(\theta))$ of $B(\theta)$ is closed in $H$ by Proposition B.3.)

Since $A$ is strictly F-differentiable at $\theta \in X$. It follows that

$$\|A(x_1) - B(\theta)x_1 - A(x_2) + B(\theta)x_2\|_X \le K_r\|x_1 - x_2\|_X \tag{3.1}$$

for all $x_1, x_2 \in B_X(\theta, r)$ with constant $K_r \to 0$ as $r \to 0$. (See the proof of [26, Cor.3]). In particular, this implies that $A$ is continuous in $B_X(\theta, r)$. Let

$$C_1 = \|(B(\theta)|_{X^\pm})^{-1}\|_{L(X^\pm, X^\pm)} \quad \text{and} \quad C_2 = \|I - P^0\|_{L(X, X^\pm)}. \tag{3.2}$$

Fix a small $r_1 > 0$ so that $C_1C_2K_{2r_1} < 1/2$. Consider the map

$$S : B_{H^0}(\theta, r_1) \times (B_X(\theta, r_1) \cap X^\pm) \to X^\pm \tag{3.3}$$

given by $S(z, x) = -(B(\theta)|_{X^\pm})^{-1}(I - P^0)A(z + x) + x$. Let $z_1, z_2 \in B_{H^0}(\theta, r_1)$ and $x_1, x_2 \in B_X(\theta, r_1) \cap X^\pm$. Noting $B(\theta)x_i \in X^\pm$ and $B(\theta)z_i = 0$, $i = 1, 2$, we get

$$\begin{aligned}
&\|S(z_1, x_1) - S(z_2, x_2)\|_{X^\pm}\\
&\le C_1 \cdot \|(I - P^0)A(z_1 + x_1) - B(\theta)x_1 - (I - P^0)A(z_2 + x_2) + B(\theta)x_2\|_{X^\pm}\\
&= C_1 \cdot \|(I - P^0)A(z_1 + x_1) - (I - P^0)B(\theta)(z_1 + x_1)\\
&\qquad\qquad -(I - P^0)A(z_2 + x_2) + (I - P^0)B(\theta)(z_2 + x_2)\|_{X^\pm}\\
&\le C_1C_2 \cdot \|A(z_1 + x_1) - B(\theta)(z_1 + x_1) - A(z_2 + x_2) + B(\theta)(z_2 + x_2)\|_X\\
&\le C_1C_2K_{2r_1} \cdot \|z_1 + x_1 - z_2 - x_2\|_X\\
&< \frac{1}{2}\|z_1 + x_1 - z_2 - x_2\|_X \quad \text{if} \quad (z_1, x_1) \ne (z_2, x_2).
\end{aligned} \tag{3.4}$$

Here the first two inequalities come from (3.2), and the third one is due to (3.1). In particular, for any $z \in B_{H^0}(\theta, r_1)$ and $x_1, x_2 \in B_X(\theta, r_1) \cap X^{\pm}$, it holds that

$$\|S(z, x_1) - S(z, x_2)\|_{X^{\pm}} < \frac{1}{2}\|x_1 - x_2\|_X \quad \text{if } x_1 \neq x_2.$$

Moreover, since $A(x) \to \theta$ as $x \to \theta$ we can choose $r_0 \in (0, r_1)$ such that

$$\begin{aligned}\|S(z, \theta)\|_{X^{\pm}} &= \|(B(\theta)|_{X^{\pm}})^{-1}(I - P^0)A(z)\|_{X^{\pm}} \\ &\leq C_1C_2\|A(z)\|_X < r_1(1 - 1/2) = \frac{r_1}{2}\end{aligned}$$

for any $z \in B_{H^0}(\theta, r_0)$. By Theorem 10.1.1 in [18, §10.1] there exists a unique map $h : B_{H^0}(\theta, r_0) \to B_X(\theta, r_1) \cap X^{\pm}$ such that $S(z, h(z)) = h(z)$ or equivalently

$$(I - P^0)A(z + h(z)) = \theta \quad \forall z \in B_{H^0}(\theta, r_0). \tag{3.5}$$

Clearly, $h(\theta) = \theta$. From this and (3.4) it follows that

$$\|h(z_1) - h(z_2)\|_X \leq 2\|z_1 - z_2\|_X \quad \forall z_1, z_2 \in B_{H^0}(\theta, r_0). \tag{3.6}$$

That is, $h$ is Lipschitz continuous.

For small $z_i \in B_{H^0}(\theta, r_0)$ set $x_i = h(z_i)$ in (3.4), $i = 1, 2$. We get

$$\begin{aligned}&\|h(z_1) - h(z_2)\|_{X^{\pm}} \\ &= \|S(z_1, h(z_1)) - S(z_2, h(z_2))\|_{X^{\pm}} \\ &\leq C_1C_2 \cdot \|A(z_1 + h(z_1)) - B(\theta)(z_1 + h(z_1)) \\ &\qquad - A(z_2 + h(z_2)) + B(\theta)(z_2 + h(z_2))\|_X.\end{aligned} \tag{3.7}$$

By (3.1), for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\|A(y_2) - A'(\theta)(y_2) - A(y_1) + A'(\theta)(y_1)\|_X \leq \varepsilon\|y_2 - y_1\|_X \tag{3.8}$$

for $y_1, y_2 \in B_X(\theta, \delta)$. Let us choose $\delta_0 \in (0, \delta)$ such that $z + h(z) \in B_X(\theta, \delta)$ for any $z \in B_{H^0}(\theta, \delta_0)$. From (3.7)- (3.8) and (3.6) it follows that

$$\|h(z_2) - h(z_1)\|_{X^{\pm}} \leq 3C_1C_2\varepsilon\|z_2 - z_1\|_X \quad \forall z_1, z_2 \in B_{H^0}(\theta, \delta_0).$$

Hence $h$ is strictly F-differentiable at $\theta \in H^0$ and $h'(\theta) = 0$.

**Step 2**. Let us prove the remainder “Moreover” part. Since $\mathcal{L}|_{V \cap X}$ is continuous and continuously directional differentiable (with respect to the induced topology on $V \cap H$ from $H$), for $z_0 \in B_{H^0}(\theta, r_0)$, $z \in H^0$ and $t \in \mathbb{R} \setminus \{0\}$ with $z_0 + tz \in B_{H^0}(\theta, r_0)$, by the mean value theorem we have $s \in (0, 1)$ such that

$$\begin{aligned}&\mathcal{L}^\circ(z_0 + tz) - \mathcal{L}^\circ(z_0) \\ =\ & D\mathcal{L}(z_{s,t})(tz + h(z_0 + tz) - h(z_0)) \\ =\ & (A(z_{s,t}), tz + h(z_0 + tz) - h(z_0))_H \\ =\ & (A(z_{s,t}), tz)_H + ((I - P^0)A(z_{s,t}), h(z_0 + tz) - h(z_0))_H\end{aligned} \tag{3.9}$$

because $h(z_0+tz)-h(z_0) \in X^\pm \subset H^\pm$, where $z_{s,t} = z_0+h(z_0)+s[tz+h(z_0+tz)-h(z_0)]$. Note that (3.6) implies

$$\|h(z_0+tz)-h(z_0)\|_H \le \|h(z_0+tz)-h(z_0)\|_X \le 2|t|r_0.$$

Let $t \to 0$, we have

$$\begin{aligned}
& \left|\frac{((I-P^0)A(z_{s,t}), h(z_0+tz)-h(z_0))_H}{t}\right| \\
\le\ & \frac{\|(I-P^0)A(z_{s,t})\|_H \cdot \|h(z_0+tz)-h(z_0)\|_H}{|t|} \\
\le\ & 2r_0\|(I-P^0)A(z_{s,t})\|_{X^\pm} \\
\to\ & 2r_0\|(I-P^0)A(z_0+h(z_0))\|_{X^\pm} = 0
\end{aligned}$$

because of (3.5) and the continuity of $A$ in $B_X(\theta, r)$. From this and (3.9) it follows that

$$D\mathcal{L}^\circ(z_0)(z) = \lim_{t\to 0}\frac{\mathcal{L}^\circ(z_0+tz)-\mathcal{L}^\circ(z_0)}{t} = (A(z_0+h(z_0)), z)_H.$$

Namely, $\mathcal{L}^\circ$ is Gâteaux differentiable at $z_0$. Clearly, $z \mapsto D\mathcal{L}^\circ(z_0)(z)$ is linear and continuous, i.e. $\mathcal{L}^\circ$ has a linear bounded Gâteaux derivative at $z_0$, $D\mathcal{L}^\circ(z_0)$, given by $D\mathcal{L}^\circ(z_0)z = (A(z_0+h(z_0)), z)_H = (P^0A(z_0+h(z_0)), z)_H\ \forall z \in H^0$.

Note that $B(\theta)|_{H^0} = 0$, $B(\theta)(H^\pm) \subset H^\pm$ and $h(z_0), h(z_0') \in X^\pm \subset H^\pm$ for any $z_0, z_0' \in B_{H^0}(\theta, r_0)$. We have

$$(P^0B(\theta)(z_0+h(z_0)), z)_H = (P^0B(\theta)(z_0'+h(z_0')), z)_H = 0 \quad \forall z \in H^0$$

From this it easily follows that

$$\begin{aligned}
& |D\mathcal{L}^\circ(z_0)z - D\mathcal{L}^\circ(z_0')z| \\
=\ & \left|\big(P^0A(z_0+h(z_0)) - P^0A(z_0'+h(z_0')), z\big)_H\right| \\
=\ & \big|\big(P^0A(z_0+h(z_0)) - P^0B(\theta)(z_0+h(z_0)), z\big)_H \\
& -\big(P^0A(z_0'+h(z_0')) - P^0B(\theta)(z_0'+h(z_0')), z\big)_H\big| \\
\le\ & \|P^0A(z_0+h(z_0)) - P^0B(\theta)(z_0+h(z_0)) \\
& -P^0A(z_0'+h(z_0')) + P^0B(\theta)(z_0'+h(z_0'))\|_H \cdot \|z\|_H \\
\le\ & \|A(z_0+h(z_0)) - B(\theta)(z_0+h(z_0)) \\
& -A(z_0'+h(z_0')) + B(\theta)(z_0'+h(z_0'))\|_H \cdot \|z\|_H \\
\le\ & \|A(z_0+h(z_0)) - B(\theta)(z_0+h(z_0)) \\
& -A(z_0'+h(z_0')) + B(\theta)(z_0'+h(z_0'))\|_X \cdot \|z\|_X \qquad (3.10) \\
\le\ & K_{r_0+r_1}\|z_0+h(z_0)-z_0'-h(z_0')\|_X \cdot \|z\|_X \\
\le\ & 3K_{r_0+r_1}\|z_0-z_0'\|_X \cdot \|z\|_X
\end{aligned}$$

because of (3.1) and (3.6). Hence $z_0 \mapsto D\mathcal{L}^\circ(z_0)$ is continuous and

$$\|D\mathcal{L}^\circ(z_0) - D\mathcal{L}^\circ(z_0')\|_{L(H^0,\mathbb{R})} \le 3K_{r_0+r_1}\|z_0 - z_0'\|_X$$

for every $z_0, z_0' \in B_X(\theta, r_0) \cap H^0$. (*Note*: Since $H$ and $X$ induce equivalent norms on $H^0$ and thus on $L(H^0, \mathbb{R})$, the alternative cannot lead to any troubles for the arguments.) By [7, Th.2.1.13], this implies that $\mathcal{L}^\circ$ is Fréchet differentiable at $z_0$ and its Fréchet differential $d\mathcal{L}^\circ(z_0) = D\mathcal{L}^\circ(z_0)$ is Lipschitz continuous in $z_0 \in B_{H^0}(\theta, r_0)$.

Now for any $\varepsilon > 0$ let $\delta > 0$ such that (3.8) holds. For $\delta_0 \in (0, \delta)$ below (3.8), by (3.10) and (3.6) we obtain

$$\begin{aligned} & |d\mathcal{L}^\circ(z_0)z - d\mathcal{L}^\circ(z_0')z| \\ \le\ & \|A(z_0 + h(z_0)) - B(\theta)(z_0 + h(z_0)) \\ & -A(z_0' + h(z_0')) + B(\theta)(z_0' + h(z_0'))\|_X \cdot \|z\|_X \\ \le\ & 3\varepsilon\|z_0 - z_0'\|_X \cdot \|z\|_X \end{aligned}$$

and hence $\|d\mathcal{L}^\circ(z_0) - d\mathcal{L}^\circ(z_0')\|_{L(H^0,\mathbb{R})} \le 3\varepsilon\|z_0 - z_0'\|_X$ for any $z_0, z_0' \in B_{H^0}(\theta, \delta_0)$. This shows that $d\mathcal{L}^\circ$ is strictly F-differentiable at $\theta \in H^0$ and $d^2\mathcal{L}^\circ(\theta) = 0$. Lemma 3.1 is proved. □

Since $\|\cdot\|$ and $\|\cdot\|_X$ are equivalent norms on $H^0$ we may choose $\delta > 0$ so small that $\bar{B}_H(\theta, \delta) \cap H^0 \subset B_X(\theta, r_0) \cap H^0$ and that

$$z + h(z) + u \in V \quad \forall (z, u) \in (\bar{B}_H(\theta, \delta) \cap H^0) \times (\bar{B}_H(\theta, \delta) \cap H^\pm). \tag{3.11}$$

**Remark 3.2.** If $A \in C^1(V^X, X)$, we can directly apply the implicit function theorem [42, Th.3.7.2] to $C^1$-map

$$T : (H^0 \cap V) \times (X^\pm \cap V) \to X^\pm,\ (z, x) \mapsto (I - P^0)A(z + x),$$

and get that the maps $h$ and $\mathcal{L}^\circ$ are $C^1$ and $C^2$, respectively. Precisely,

$$h'(z) = -\big[(I - P^0)A'(z + h(z))|_{X^\pm}\big]^{-1}(I - P^0)A'(z + h(z))|_{H^0}.$$

□

Define a continuous map $F : \bar{B}_{H^0}(\theta, \delta) \times B_{H^\pm}(\theta, \delta) \to \mathbb{R}$ as

$$F(z, u) = \mathcal{L}(z + h(z) + u) - \mathcal{L}(z + h(z)). \tag{3.12}$$

Then for each $z \in \bar{B}_{H^0}(\theta, \delta)$ the map $F(z, \cdot)$ is continuously directional differentiable in $B_{H^\pm}(\theta, \delta)$, and the directional derivative of it at $u \in B_{H^\pm}(\theta, \delta)$ in any direction $v \in H^\pm$ is given by

$$\begin{aligned} D_2F(z, u)(v) &= (A(z + h(z) + u), v)_H \\ &= ((I - P^0)A(z + h(z) + u), v)_H. \end{aligned} \tag{3.13}$$

It follows from this and (3.5) that

$$F(z,\theta)=0 \quad \text{and} \quad D_2F(z,\theta)(v)=0\ \forall v\in H^{\pm}. \tag{3.14}$$

Now we wish to apply Theorem A.1 to the function $F$. In order to check that $F$ satisfies the conditions in Theorem A.1 we need two lemmas.

**Lemma 3.3.** *There exists a function* $\omega : V\cap X \to [0,\infty)$ *such that* $\omega(x)\to 0$ *as* $x\in V\cap X$ *and* $\|x\|\to 0$*, and that*

$$|(B(x)u,v)_H-(B(\theta)u,v)_H|\le \omega(x)\|u\|\cdot\|v\|$$

*for any* $x\in V\cap X$*,* $u\in H^0\oplus H^-$ *and* $v\in H$*.*

*Proof.* Note that the condition (D2) can be equivalently expressed as: *For any* $u\in H$ *it holds that* $\|P(x)u-P(\theta)u\|\to 0$ *as* $x\in V\cap X$ *and* $\|x\|\to 0$. Let $e_1,\cdots,e_m$ be a basis of $H^0\oplus H^-$ with $\|e_i\|=1$, $i=1,\cdots,m$. Then

$$\begin{aligned}
&|(B(x)e_i,v)_H-(B(\theta)e_i,v)_H|\\
\le\ & |(P(x)e_i-P(\theta)e_i,v)_H|+|([Q(x)-Q(\theta)]e_i,v)_H|\\
\le\ & \|P(x)e_i-P(\theta)e_i\|\cdot\|v\|+\|Q(x)-Q(\theta)\|\cdot\|v\|.
\end{aligned}$$

Since $H^0\oplus H^-$ is of finite dimension, there exists a constant $C_4>0$ such that

$$\left(\sum_{i=1}^m|t_i|^2\right)^{1/2}\le C_4\|u\| \quad \forall u=\sum_{i=1}^m t_ie_i\in H^0\oplus H^-.$$

Hence for any $u=\sum_{i=1}^m t_ie_i\in H^0\oplus H^-$ we have

$$\begin{aligned}
&|(B(x)u,v)_H-(B(\theta)u,v)_H|\\
\le\ & \sum_{i=1}^m|t_i|\|P(x)e_i-P(\theta)e_i\|\cdot\|v\|+\sum_{i=1}^m|t_i|\|Q(x)-Q(\theta)\|\cdot\|v\|\\
\le\ & \left(\sum_{i=1}^m\|P(x)e_i-P(\theta)e_i\|^2\right)^{1/2}\left(\sum_{i=1}^m|t_i|^2\right)^{1/2}\|v\|\\
&+\sqrt{m}\left(\sum_{i=1}^m|t_i|^2\right)^{1/2}\|Q(x)-Q(\theta)\|\cdot\|v\|\\
\le\ & \left[C_4\left(\sum_{i=1}^m\|P(x)e_i-P(\theta)e_i\|^2\right)^{1/2}+C_4\sqrt{m}\|Q(x)-Q(\theta)\|\right]\|u\|\|v\|\\
=\ & \omega(x)\|u\|\|v\|,
\end{aligned}$$

where

$$\omega(x) = \left[ C_4 \left( \sum_{i=1}^{m} \|P(x)e_i - P(\theta)e_i\|^2 \right)^{1/2} + C_4\sqrt{m}\|Q(x) - Q(\theta)\| \right] \to 0$$

as $x \in V \cap X$ and $\|x\| \to 0$ (because of the conditions (D2) and (D3)). □

When $H^0 = \{\theta\}$ under the stronger assumptions the following lemma was proved in [43, 46]. We also give proof of it for clearness.

**Lemma 3.4.** *There exists a small neighborhood $U \subset V$ of $\theta$ in $H$ and a number $a_1 \in (0, 2a_0]$ such that for any $x \in U \cap X$,*

(i) $(B(x)u, u)_H \ge a_1\|u\|^2 \ \forall u \in H^+$*;*

(ii) $|(B(x)u, v)_H| \le \omega(x)\|u\| \cdot \|v\| \ \forall u \in H^+, \forall v \in H^- \oplus H^0$*;*

(iii) $(B(x)u, u)_H \le -a_0\|u\|^2 \ \forall u \in H^-$*.*

*Proof.* **(i)** By (2.2), we have

$$(B(\theta)u, u)_H \ge 2a_0\|u\|^2 \ \forall u \in H^+. \tag{3.15}$$

Assume by contradiction that (i) does not hold. Then there exist sequences $\{x_n\} \subset V \cap X$ with $\|x_n\| \to 0$, and $\{u_n\} \in H^+$ with $\|u_n\| = 1 \ \forall n$, such that

$$(B(x_n)u_n, u_n)_H < 1/n \ \forall n = 1, 2, \cdots.$$

Passing a subsequence, we may assume that

$$(B(x_n)u_n, u_n)_H \to \beta \le 0 \text{ as } n \to \infty, \tag{3.16}$$

and that $u_n \rightharpoonup u_0$ in $H$. We claim: $u_0 \ne \theta$. In fact, by the condition (D4) we have constants $C_0 > 0$ and $n_0 \in \mathbb{N}$ such that $(P(x_n)u, u) \ge C_0\|u\|^2$ for any $u \in H$ and $n \ge n_0$. Hence

$$\begin{aligned}(B(x_n)u_n, u_n)_H &= (P(x_n)u_n, u_n)_H + (Q(x_n)u_n, u_n)_H \\ &\ge C_0 + (Q(x_n)u_n, u_n)_H \quad \forall n > n_0.\end{aligned} \tag{3.17}$$

Moreover, a direct computation gives

$$\begin{aligned}&|(Q(x_n)u_n, u_n)_H - (Q(\theta)u_0, u_0)_H| \\ &= |((Q(x_n) - Q(\theta))u_n, u_n)_H + (Q(\theta)u_n, u_n)_H - (Q(\theta)u_0, u_n)_H \\ &\qquad + (Q(\theta)u_0, u_n - u_0)_H| \\ &\le \|Q(x_n) - Q(\theta)\| \cdot \|u_n\|^2 + \|Q(\theta)u_n - Q(\theta)u_0\| \cdot \|u_n\| \\ &\qquad + |(Q(\theta)u_0, u_n - u_0)_H| \\ &\le \|Q(x_n) - Q(\theta)\| + \|Q(\theta)u_n - Q(\theta)u_0\| + |(Q(\theta)u_0, u_n - u_0)_H|.\end{aligned} \tag{3.18}$$

Since $u_n \rightharpoonup u_0$ in $H$, $\lim_{n\to\infty} |(Q(\theta)u_0, u_n - u_0)_H| = 0$. We have also

$$\lim_{n\to\infty} \|Q(\theta)u_n - Q(\theta)u_0\| = 0 \tag{3.19}$$

by the compactness of $Q(\theta)$, and

$$\lim_{n\to\infty} \|Q(x_n) - Q(\theta)\| = 0 \tag{3.20}$$

by the condition (D3). Hence (3.18)-(3.20) give

$$\lim_{n\to\infty} (Q(x_n)u_n, u_n)_H = (Q(\theta)u_0, u_0)_H. \tag{3.21}$$

Then this and (3.16)-(3.17) yield

$$0 \geq \beta = \lim_{n\to\infty} (B(x_n)u_n, u_n)_H \geq C_0 + (Q(\theta)u_0, u_0)_H.$$

This implies $u_0 \neq \theta$. Note that $u_0$ also sits in $H^+$.

As above, using (3.20) we derive

$$\begin{aligned} & |(Q(x_n)u_0, u_n)_H - (Q(\theta)u_0, u_0)_H| \qquad (3.22) \\ \leq\ & |(Q(x_n)u_0, u_n)_H - (Q(\theta)u_0, u_n)_H| + |(Q(\theta)u_0, u_n)_H - (Q(\theta)u_0, u_0)_H| \\ \leq\ & \|Q(x_n) - Q(\theta)\| \cdot \|u_0\| + |(Q(\theta)u_0, u_n - u_0)_H| \to 0. \end{aligned}$$

Note that

$$\begin{aligned} & (B(x_n)(u_n - u_0), u_n - u_0)_H \\ =\ & (P(x_n)(u_n - u_0), u_n - u_0)_H + (Q(x_n)(u_n - u_0), u_n - u_0)_H \\ \geq\ & C_0\|u_n - u_0\|^2 + (Q(x_n)(u_n - u_0), u_n - u_0)_H \\ \geq\ & (Q(x_n)u_n, u_n)_H - 2(Q(x_n)u_0, u_n)_H + (Q(\theta)u_0, u_0)_H. \end{aligned}$$

It follows from this and (3.21)-(3.22) that

$$\begin{aligned} & \liminf_{n\to\infty} (B(x_n)(u_n - u_0), u_n - u_0)_H \\ & \geq \lim_{n\to\infty} (Q(x_n)(u_n - u_0), u_n - u_0)_H = 0. \end{aligned} \tag{3.23}$$

Note that $u_n \rightharpoonup u_0$ implies that $(P(\theta)u_0, u_n - u_0)_H \to 0$. We get

$$\begin{aligned} & |(B(x_n)u_0, u_n)_H - (B(\theta)u_0, u_0)_H| \\ =\ & |(P(x_n)u_0, u_n)_H + (Q(x_n)u_0, u_n)_H - (P(\theta)u_0, u_0)_H - (Q(\theta)u_0, u_0)_H| \\ \leq\ & |(P(x_n)u_0, u_n)_H - (P(\theta)u_0, u_0)_H| + |(Q(x_n)u_0, u_n)_H - (Q(\theta)u_0, u_0)_H| \\ \leq\ & |(P(x_n)u_0, u_n)_H - (P(\theta)u_0, u_n)_H| + |(P(\theta)u_0, u_n)_H - (P(\theta)u_0, u_0)_H| \\ & +|(Q(x_n)u_0, u_n)_H - (Q(\theta)u_0, u_0)_H| \\ \leq\ & \|P(x_n)u_0 - P(\theta)u_0\| + |(P(\theta)u_0, u_n - u_0)_H| \\ & +|(Q(x_n)u_0, u_n)_H - (Q(\theta)u_0, u_0)_H| \to 0 \end{aligned}$$

because of the condition (D2) and (3.22). Similarly, we have

$$\lim_{n\to\infty}(B(x_n)u_0, u_0)_H = (B(\theta)u_0, u_0)_H.$$

From these, (3.16) and (3.23) it follows that

$$\begin{aligned}
0 &\le \liminf_{n\to\infty}(B(x_n)(u_n-u_0), u_n-u_0)_H \\
&= \liminf_{n\to\infty}[(B(x_n)u_n, u_n)_H - 2(B(x_n)u_0, u_n)_H + (B(x_n)u_0, u_0)_H] \\
&= \lim_{n\to\infty}(B(x_n)u_n, u_n)_H - (B(\theta)u_0, u_0)_H \\
&= \beta - (B(\theta)u_0, u_0)_H.
\end{aligned}$$

Namely, $(B(\theta)u_0, u_0)_H \le \beta \le 0$. It contradicts to (3.15) because $u_0 \in H^+ \setminus \{0\}$.
**(ii)** By (2.1), $(B(\theta)u, v)_H = 0$ for $u \in H^+$ and $v \in H^0 \oplus H^-$. The conclusion follows from Lemma 3.3 immediately.
**(iii)** By the choice of $a_0$ we have $(B(\theta)v, v)_H \le -2a_0\|v\|^2\ \forall v \in H^-$. By Lemma 3.3, for any $x \in U \cap X$ and $v \in H^-$ we have

$$\begin{aligned}
(B(x)v, v)_H &= (B(\theta)v, v)_H + (B(x)v, v)_H - (B(\theta)v, v)_H \\
&\le (B(\theta)v, v)_H + \omega(x)\|v\|^2 \\
&\le -2a_0\|v\|^2 + \omega(x)\|v\|^2.
\end{aligned}$$

By shrinking $U$ (if necessary) we can require that $\omega(x) < a_0$ for any $x \in U \cap X$. Then the desired conclusion is proved. □

Since $h(\theta) = \theta$, for the neighborhood $U$ in Lemma 3.4 we may take $\varepsilon \in (0, \delta)$ so small that

$$z + h(z) + u^+ + u^- \in U \tag{3.24}$$

for all $z \in \bar{B}_{H^0}(\theta, \varepsilon)$, $u^+ \in \bar{B}_{H^+}(\theta, \varepsilon)$ and $u^- \in \bar{B}_{H^-}(\theta, \varepsilon)$.

**Lemma 3.5.** *For the above $\varepsilon > 0$ the restriction of the function $F$ in (3.14) to $\bar{B}_{H^0}(\theta, \varepsilon) \times \big(\bar{B}_{H^+}(\theta, \varepsilon) \oplus \bar{B}_{H^-}(\theta, \varepsilon)\big)$ satisfies the conditions in Theorem A.1.*

*Proof*. By (3.12) we only need to prove that $F$ satisfies conditions (ii)-(iv) in Theorem A.1.

**Step 1**. For $z \in \bar{B}_{H^0}(\theta, \varepsilon)$, $u^+ \in \bar{B}_H(\theta, \varepsilon) \cap X^+$ and $u_1^-, u_2^- \in \bar{B}_{H^-}(\theta, \varepsilon)$, by the condition (F2) we have

$$\begin{aligned}
&[D_2F(z, u^+ + u_2^-) - D_2F(z, u^+ + u_1^-)](u_2^- - u_1^-) \\
= \;& (A(z + h(z) + u^+ + u_2^-), u_2^- - u_1^-)_H \\
&-(A(z + h(z) + u^+ + u_1^-), u_2^- - u_1^-)_H.
\end{aligned}$$

Moreover, $A$ is continuously directional differentiable so is the function

$$u \mapsto (A(z + h(z) + u^+ + u), u_2^- - u_1^-)_H.$$

By the mean value theorem we have $t \in (0, 1)$ such that

$$\begin{aligned} & (A(z + h(z) + u^+ + u_2^-), u_2^- - u_1^-)_H \\ & -(A(z + h(z) + u^+ + u_1^-), u_2^- - u_1^-)_H \\ = & \big(DA(z + h(z) + u^+ + u_1^- + t(u_2^- - u_1^-))(u_2^- - u_1^-), u_2^- - u_1^-\big)_H \\ \overset{(F3)}{=} & \big(B(z + h(z) + u^+ + u_1^- + t(u_2^- - u_1^-))(u_2^- - u_1^-), u_2^- - u_1^-\big)_H \\ \leq & -a_0\|u_2^- - u_1^-\|^2 \end{aligned}$$

by Lemma 3.4(iii). Hence

$$[D_2F(z, u^+ + u_2^-) - D_2F(z, u^+ + u_1^-)](u_2^- - u_1^-) \leq -a_0\|u_2^- - u_1^-\|^2.$$

Since $\bar{B}_H(\theta, \varepsilon) \cap X^+$ is dense in $\bar{B}_H(\theta, \varepsilon) \cap H^+$ we get

$$[D_2F(z, u^+ + u_2^-) - D_2F(z, u^+ + u_1^-)](u_2^- - u_1^-) \leq -a_0\|u_2^- - u_1^-\|^2. \tag{3.25}$$

for all $z \in \bar{B}_{H^0}(\theta, \varepsilon)$, $u^+ \in \bar{B}_H(\theta, \varepsilon) \cap H^+$ and $u_i^- \in \bar{B}_H(\theta, \varepsilon) \cap H^-$, $i = 1, 2$. This implies the condition (ii).

**Step 2**. Let $z \in \bar{B}_{H^0}(\theta, \varepsilon)$, $u^+ \in \bar{B}_H(\theta, \varepsilon) \cap X^+$ and $u^- \in \bar{B}_{H^-}(\theta, \varepsilon)$. Then by (3.14), the mean value theorem and (F2)-(F3), for some $t \in (0, 1)$ we have

$$\begin{aligned} & D_2F(z, u^+ + u^-)(u^+ - u^-) \\ = \; & D_2F(z, u^+ + u^-)(u^+ - u^-) - D_2F(z, \theta)(u^+ - u^-) \\ = \; & (A(z + h(z) + u^+ + u^-), u^+ - u^-)_H - (A(z + h(z) + \theta), u^+ - u^-)_H \\ = \; & \big(B(z + h(z) + t(u^+ + u^-))(u^+ + u^-), u^+ - u^-\big)_H \\ = \; & \big(B(z + h(z) + t(u^+ + u^-))u^+, u^+\big)_H \\ - \; & \big(B(z + h(z) + t(u^+ + u^-))u^-, u^-\big)_H \\ \geq \; & a_1\|u^+\|^2 + a_0\|u^-\|^2 \end{aligned}$$

by Lemma 3.4(i) and (iii). As above this inequality also holds for all $u^+ \in \bar{B}_{H^+}(\theta, \varepsilon)$ because $\bar{B}_H(\theta, \varepsilon) \cap X^+$ is dense in $\bar{B}_H(\theta, \varepsilon) \cap H^+$. Hence $D_2F(z, u^+ + u^-)(u^+ - u^-) > 0$ for $(u^+, u^-) \neq (\theta, \theta)$. The condition (iii) is proved.

**Step 3**. For $z \in \bar{B}_{H^0}(\theta, \varepsilon)$ and $u^+ \in \bar{B}_H(\theta, \varepsilon) \cap X^+$, as above we have $t \in (0, 1)$ such that

$$\begin{aligned} D_2F(z, u^+)u^+ &= D_2F(z, u^+)u^+ - D_2F(z, \theta)u^+ \\ &= (A(z + h(z) + u^+), u^+)_H - (A(z + h(z) + \theta), u^+)_H \\ &= \big(B(z + h(z) + tu^+)u^+, u^+\big)_H \\ &\geq a_1\|u^+\|^2 \end{aligned}$$

because of Lemma 3.4(i). It follows that

$$D_2F(z,u^+)u^+ \ge a_1\|u^+\|^2 > p(\|u^+\|) \quad \forall u^+ \in \bar{B}_H(\theta,\varepsilon)\cap H^+\setminus\{\theta\},$$

where $p:(0,\varepsilon]\to(0,\infty)$ is a non-decreasing function given by $p(t)=\frac{a_1}{2}t^2$. This proves the condition (iv). $\square$

By Lemma 3.5 we can apply Theorem A.1 to $F$ to get a positive number $\epsilon$, an open neighborhood $\mathcal{W}$ of $\bar{B}_{H^0}(\theta,\varepsilon)\times\{\theta\}$ in $\bar{B}_{H^0}(\theta,\varepsilon)\times H^\pm$, and an origin-preserving homeomorphism

$$\phi:\bar{B}_{H^0}(\theta,\varepsilon)\times(B_{H^+}(\theta,\epsilon)+B_{H^-}(\theta,\epsilon))\to\mathcal{W} \tag{3.26}$$

of form

$$\phi(z,u^++u^-)=(z,\phi_z(u^++u^-))\in\bar{B}_{H^0}(\theta,\varepsilon)\times H^\pm$$

such that $\phi_z(\theta)=\theta$ and

$$\begin{aligned}&\mathcal{L}(z+h(z)+\phi_z(u^+,u^-))-\mathcal{L}(z+h(z))\\ =\;& F(\phi(z,u^+,u^-))=\|u^+\|^2-\|u^-\|^2\end{aligned} \tag{3.27}$$

for all $(z,u^+,u^-)\in\bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^+}(\theta,\epsilon)\times B_{H^-}(\theta,\epsilon)$. Moreover, $\phi_z(u^++u^-)\in H^-$ if and only if $u^+=\theta$, and $\phi$ is also a homeomorphism from $\bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^-}(\theta,\epsilon)$ onto $\mathcal{W}\cap(\bar{B}_{H^0}(\theta,\varepsilon)\times H^-)$ even if the last two sets are equipped with the induced topology from $X$, or, equivalently, for $(z_0,u_0^-)\in\bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^-}(\theta,\epsilon)$ and $\{(z_k,u_k^-)\}\subset\bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^-}(\theta,\epsilon)$ it holds that

$$\|z_k+u_k^--z_0-u_0^-\|_X\to 0\Longleftrightarrow\left\{\begin{array}{l}\|z_k-z_0\|_X\to 0\quad\text{and}\\ \|\phi_{z_k}(u_k^-)-\phi_{z_0}(u_0^-)\|_X\to 0.\end{array}\right. \tag{3.28}$$

Consider the continuous map

$$\begin{aligned}\Phi: B_{H^0}(\theta,\varepsilon)\times(B_{H^+}(\theta,\epsilon)+B_{H^-}(\theta,\epsilon))&\to H,\\ (z,u^++u^-)&\mapsto z+h(z)+\phi_z(u^++u^-).\end{aligned} \tag{3.29}$$

Then (3.27) gives (2.5), i.e. $\mathcal{L}(\Phi(z,u^+,u^-))=\|u^+\|^2-\|u^-\|^2+\mathcal{L}(z+h(z))$. Since $H^0$ and $H^-$ are finitely dimensional subspaces contained in $X$, from Steps 1,4 in the proof of Theorem A.1 it is easily seen that

$$\phi_z\left(B_{H^+}(\theta,\epsilon)\cap X+B_{H^-}(\theta,\epsilon)\right)\subset X\quad\forall z\in B_{H^0}(\theta,\varepsilon).$$

Then (2.6) follows from this and the fact that $\mathrm{Im}(h)\subset X^\pm\subset X$. In particular, it holds that $\Phi(B_{H^0}(\theta,\varepsilon)\times B_{H^-}(\theta,\epsilon))\subset X$. Now we can complete the proof of Theorem 2.1 by the following lemma.

**Lemma 3.6.** *Let $W = \mathrm{Im}(\Phi)$. Then it is an open neighborhood of $\theta$ in $H$ and $\Phi$ is an origin-preserving homeomorphism onto $W$. Moreover, if the topologies on $B_{H^0}(\theta,\varepsilon) \times B_{H^-}(\theta,\epsilon) \subset X$ and $\Phi(B_{H^0}(\theta,\varepsilon) \times B_{H^-}(\theta,\epsilon)) \subset X$ are chosen as ones induced by $X$, the restriction of $\Phi$ to $B_{H^0}(\theta,\varepsilon) \times B_{H^-}(\theta,\epsilon)$ is a homeomorphism from $B_{H^0}(\theta,\varepsilon) \times B_{H^-}(\theta,\epsilon) \subset X$ onto $\Phi(B_{H^0}(\theta,\varepsilon) \times B_{H^-}(\theta,\epsilon)) \subset X$.*

*Proof.* Assume that $\Phi(z_1, u_1^+ + u_1^-) = \Phi(z_2, u_2^+ + u_2^-)$ for $(z_1, u_1^+ + u_1^-)$ and $(z_2, u_2^+ + u_2^-)$ in $B_{H^0}(\theta,\varepsilon) \times (B_{H^+}(\theta,\epsilon) + B_{H^-}(\theta,\epsilon))$. Then

$$z_1 = z_2 \quad \text{and} \quad h(z_1) + \phi_{z_1}(u_1^+ + u_1^-) = h(z_2) + \phi_{z_2}(u_2^+ + u_2^-).$$

It follows that $h(z_1) = h(z_2)$ and $\phi_{z_1}(u_1^+ + u_1^-) = \phi_{z_2}(u_2^+ + u_2^-)$. This shows that $\phi(z_1, u_1^+ + u_1^-) = \phi(z_2, u_2^+ + u_2^-)$ and thus $(u_1^+, u_1^-) = (u_2^+, u_2^-)$. So $\Phi$ is a bijection.

Let $(z, u^+ + u^-)$ and a sequence $\{(z_k, u_k^+ + u_k^-)\}$ sit in $B_{H^0}(\theta,\varepsilon) \times (B_{H^+}(\theta,\epsilon) + B_{H^-}(\theta,\epsilon))$. Suppose that $\Phi(z_k, u_k^+ + u_k^-) \to \Phi(z, u^+ + u^-)$. Then

$$\begin{aligned} &P^0\Phi(z_k, u_k^+ + u_k^-) \to P^0\Phi(z, u^+ + u^-) \quad \text{and} \\ &(P^+ + P^-)\Phi(z_k, u_k^+ + u_k^-) \to (P^+ + P^-)\Phi(z, u^+ + u^-). \end{aligned}$$

It follows that $z_k \to z$, and thus $h(z_k) \to h(z)$ and $\phi_{z_k}(u_k^+ + u_k^-) \to \phi_z(u^+ + u^-)$. This shows that $\phi(z_k, u_k^+ + u_k^-) \to \phi(z, u^+ + u^-)$ and hence $(z_k, u_k^+ + u_k^-) \to (z, u^+ + u^-)$ since $\phi$ is a homeomorphism. That is, $\Phi^{-1}$ is continuous. The first claim is proved.

To prove the second claim, it suffices to prove that for $(z_0, u_0^-) \in \bar{B}_{H^0}(\theta,\varepsilon) \times B_{H^-}(\theta,\epsilon)$ and $\{(z_k, u_k^-)\} \subset \bar{B}_{H^0}(\theta,\varepsilon) \times B_{H^-}(\theta,\epsilon)$

$$\left.\begin{array}{l} \|z_k + u_k^- - z_0 - u_0^-\|_X \to 0 \quad \text{if and only if} \\ \|z_k + h(z_k) + \phi_{z_k}(u_k^-) - z_0 - h(z_0) - \phi_{z_0}(u_0^-)\|_X \to 0. \end{array}\right\} \tag{3.30}$$

Note that $h \in C(B_{H^0}(\theta,\delta), X^{\pm})$ and that $X$ and $H$ induce equivalent topologies on $H^0 + H^-$. Since $\|z_k + u_k^- - z_0 - u_0^-\|_X \to 0$ if and only if $\|z_k - z_0\|_X \to 0$ and $\|u_k^- - u_0^-\|_X \to 0$, it follows from (3.28) that in (3.30) the left side implies the right side. Conversely, if the right of (3.30) holds, then

$$\begin{aligned} \|z_k - z_0\| &= \|P^0(z_k + h(z_k) + \phi_{z_k}(u_k^-)) - P^0(z_0 + h(z_0) + \phi_{z_0}(u_0^-))\| \\ &= \|z_k + h(z_k) + \phi_{z_k}(u_k^-) - z_0 - h(z_0) - \phi_{z_0}(u_0^-)\| \\ &\le \|z_k + h(z_k) + \phi_{z_k}(u_k^-) - z_0 - h(z_0) - \phi_{z_0}(u_0^-)\|_X \to 0, \end{aligned}$$

and hence $\|z_k - z_0\|_X \to 0$. It follows that $\|h(z_k) - h(z_0)\|_X \to 0$ and therefore

$$\begin{aligned} &\|\phi_{z_k}(u_k^-) - \phi_{z_0}(u_0^-)\|_X \\ \le\ &\|z_k - z_0\|_X + \|h(z_k) - h(z_0)\|_X \\ &+ \|z_k + h(z_k) + \phi_{z_k}(u_k^-) - z_0 - h(z_0) - \phi_{z_0}(u_0^-)\|_X \to 0. \end{aligned}$$

From these and (3.28) we derive that $\|z_k + u_k^- - z_0 - u_0^-\|_X \to 0$. (3.30) is proved. $\square$

In summary we have completed the proof of Theorem 2.1.

# 4 Proofs of Corollaries 2.5, 2.7 and Theorem 2.10

## 4.1 Proof of Corollaries 2.5 and 2.7

*Proof of Corollary 2.5.* By the excision property of relative homology groups we only need to prove the corollary for some open neighborhood $W$ of $\theta$ in $H$. Let $W$ be as in Theorem 2.1, that is,

$$W = \Phi\left(B_{H^0}(\theta,\epsilon) \times (B_{H^+}(\theta,\epsilon) + B_{H^-}(\theta,\epsilon))\right).$$

Set $W_{0-} := \Phi\left(B_{H^0}(\theta,\epsilon) \times B_{H^-}(\theta,\epsilon)\right)$. It is contained in $X$ by (2.6). We write $W_{0-}$ as $W_{0-}^X$ when it is considered a topological subspace of $X$. Clearly, $\mathcal{L}_0 \cap W_{0-} = (\mathcal{L}|_{V\cap X})_0 \cap W_{0-}^X = \mathcal{L}_0 \cap W_{0-}^X$ as sets. Define a deformation $\eta : W \times [0,1] \to W$ as

$$\eta(\Phi(z, u^+ + u^-), t) = \Phi(z, tu^+ + u^-).$$

It gives a deformation retract from $\mathcal{L}_0 \cap W$ onto $\mathcal{L}_0 \cap W_{0-}$. Hence the inclusion

$$I : (\mathcal{L}_0 \cap W_{0-}, \mathcal{L}_0 \cap W_{0-} \setminus \{\theta\}) \hookrightarrow (\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\})$$

induces isomorphisms between their relative singular homology groups with inverse $(\eta_1)_*$, where $\eta_1(\cdot) = \eta(1,\cdot)$. That means that each

$$\alpha \in H_q\left(\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}; \mathbf{K}\right)$$

has a relative singular cycle representative, $c = \sum_j g_j\sigma_j$, such that

$$|c| := \cup_j \sigma_j(\triangle^q) \subset \mathcal{L}_0 \cap W_{0-} \quad \text{and} \quad |\partial c| \subset \mathcal{L}_0 \cap W_{0-} \setminus \{\theta\}.$$

By the conclusion (b) in Theorem 2.1 the identity map

$$\imath^{0-} : \left(\mathcal{L}_0 \cap W_{0-}^X, \mathcal{L}_0 \cap W_{0-}^X \setminus \{\theta\}\right) \to (\mathcal{L}_0 \cap W_{0-}, \mathcal{L}_0 \cap W_{0-} \setminus \{\theta\})$$

is a homeomorphism. So $c$ is also a relative singular cycle in

$$\left(\mathcal{L}_0 \cap W_{0-}^X, \mathcal{L}_0 \cap W_{0-}^X \setminus \{\theta\}\right),$$

denoted by $c^x$. Then $\imath^{0-} \circ c^x = c$. Denote by the inclusion

$$\jmath : \left(\mathcal{L}_0 \cap W^X, \mathcal{L}_0 \cap W^X \setminus \{\theta\}\right) \hookrightarrow (\mathcal{L}_0 \cap W, \mathcal{L}_0 \cap W \setminus \{\theta\}).$$

Write $W^X = W \cap X$ as a topological subspace of $X$, and denote by the inclusion

$$I^X : \left(\mathcal{L}_0 \cap W_{0-}^X, \mathcal{L}_0 \cap W_{0-}^X \setminus \{\theta\}\right) \hookrightarrow \left(\mathcal{L}_0 \cap W^X, \mathcal{L}_0 \cap W^X \setminus \{\theta\}\right).$$

Since $I_*([c]) = \alpha$, $(\imath^{0-})_*([c^x]) = [c]$ and $I \circ \imath^{0-} = \jmath \circ I^X$ we obtain

$$\alpha = I_* \circ (\imath^{0-})_*[c^x] = \jmath_* \circ (I^X)_*[c^x] = \jmath_*\big((I^X)_*[c^x]\big).$$

This completes the proof of Corollary 2.5. $\square$

*Proof of Corollary 2.7.* As in the proof of [3, Prop.3.2] we only need to prove the implication (iii)$\Longrightarrow$(i). If $\nu = \dim H^0 = 0$, by (i) of Remark 2.2 and (2.7) we have $C_q(\mathcal{L}, \theta; \mathbb{K}) = \delta_{q\mu}\ \forall q \in \mathbb{Z}$, where $\mu = \dim H^-$. Hence $\mu = 0$. Then (2.7) shows that $\theta$ is a strict minimum. If $\nu > 0$, by Corollary 2.6 it must hold that $\mu = \dim H^- = 0$ and $C_0(\mathcal{L}^\circ, \theta; \mathbb{K}) \neq 0$. Since $\mathcal{L}^\circ$ is $C^{2-0}$ and $\dim H^0 < \infty$ we can construct a $C^{2-0}$ function $g$ on $H^0$ satisfying (PS) such that it coincides with $\mathcal{L}^\circ$ near $\theta \in H^0$. By Theorem 4.6 on the page 43 of [11], $\theta$ is a minimum of $\mathcal{L}^\circ$. It follows from (2.5) that $\theta$ is a strict minimum of $\mathcal{L}$. □

## 4.2 Proofs of Theorem 2.10

Recall that $H^0 = \mathrm{Ker}(B(\theta))$ and $X^\pm = X \cap H^\pm = (I_H - P^0)(X)$. Set $Y^\pm = Y \cap H^\pm = (I_H - P^0)(Y)$. We need the following theorem by Ming Jiang.

**Theorem 4.1** ([27, Th.2.5])**.** *Under the assumptions of Theorem 2.10, (but it suffices to assume the density of $X$ in $Y$), there exists a ball $B_Y(\theta, \kappa)$, an origin-preserving local homeomorphism $\Psi$ defined on $B_Y(\theta, \kappa)$ and a $C^1$ map $\rho : B_{H^0}(\theta, \kappa) \cap H^0 \to X^\pm$ such that*

$$\mathcal{L} \circ \Psi(y) = \frac{1}{2}(B(\theta)y^\pm, y^\pm)_H + \mathcal{L}(z + \rho(z)) \quad \forall y \in B_{H^0}(\theta, \kappa),$$

*where $z = P^0(y)$ and $y^\pm = (I - P^0)(y)$. Moreover, $\Psi(B_Y(\theta, \kappa) \cap X) \subset X$ and $\Psi : B_Y(\theta, \kappa) \cap X \to \Psi(B_Y(\theta, \kappa) \cap X)$ is also an origin-preserving local homeomorphism even if both $B_Y(\theta, \kappa) \cap X$ and $\Psi(B_Y(\theta, \kappa) \cap X)$ are equipped with the induced topology by $X$.*

**Remark 4.2.** (i) From the arguments of Lemma 3.1 and the proof of [27] it is easily seen that near $\theta \in N$ the map $\rho$ is equal to $h$ in Lemma 3.1.
(ii) It was proved in [27, Prop.2.1] that the condition (iii) in Theorem 2.10 can be derived from others of this proposition and the following two conditions:

**(FN3a)** $\forall x \in V \cap X$, $\exists\, C(x) > 0$ such that

$$|d^2(\mathcal{L}|_{V^X})(x)(\xi, \eta)| \le C(x)\|\xi\| \cdot \|\eta\|\ \forall \xi, \eta \in X.$$

**(FN3b)** $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that for any $x_1, x_2 \in V \cap X$ with $\|x_1 - x_2\|_Y < \delta$,

$$|d^2(\mathcal{L}|_{V^X})(x_1)(\xi, \eta) - d^2(\mathcal{L}|_{V^X})(x_2)(\xi, \eta)| \le \varepsilon\|\xi\| \cdot \|\eta\|\ \forall \xi, \eta \in X.$$

If $H^- \subset Y$, then $P^+Y \subset Y$ because $H^0 \subset X \subset Y$. In this case, for $y \in Y$ we can write $y^\perp = (I - P^0)y = y^+ + y^- = P^+y + P^-y$ and hence

$$(B(\theta)y^\perp, y^\perp)_H = (P^+B(\theta)P^+y^+, y^+)_H + (P^-B(\theta)P^-y^-, y^-)_H$$

Define a functional $\mathcal{L}^\diamond : B_{H^0}(\theta,\kappa)\cap H^0\to\mathbb{R}$ by $\mathcal{L}^\diamond(z)=\mathcal{L}(z+\rho(z))$. Then $\theta\in H^0$ is its critical point, and also isolated if $\theta$ is an isolated critical point of $\mathcal{L}|_{V^X}$. By Remark 3.2, $\rho$ is $C^1$, and Lemma 3.1 and Remark 4.2(i) show that near $\theta\in H^0$,

$$d\mathcal{L}^\diamond(z)(\xi)=(A(z+\rho(z)),\xi)_H=(A(z+h(z)),\xi)_H\quad\forall\xi\in H^0.$$

If $\theta$ is an isolated critical point of $\mathcal{L}|_{V^X}$ (and hence $\mathcal{L}|_{V^Y}$), then by Theorem 4.1 we can use the same proof method as in [37, Th.8.4] or [12, Th.5.1.17] to derive:

**Corollary 4.3** (Shifting)**.** *Under the assumptions of Theorem 4.1, if $\theta$ is an isolated critical point of $\mathcal{L}|_{V^Y}$, $H^-\subset Y$ and $\dim H^0\oplus H^-<\infty$, then*

$$C_q(\mathcal{L}|_{V^Y},\theta;\mathbf{K})\cong C_{q-\mu}(\mathcal{L}^\diamond,\theta;\mathbf{K})\quad\forall q\in\mathbb{N}\cup\{0\}$$

*for any Abel group $\mathbf{K}$, where $\mu:=\dim H^-$.*

**Corollary 4.4** ([27, Cor.2.8])**.** *Under the assumptions of Theorem 4.1, if $\theta$ is an isolated critical point of $\mathcal{L}|_{V^Y}$, and $H^-\subset X$, then for any Abel group $\mathbf{K}$,*

$$C_q(\mathcal{L}|_{V^X},\theta;\mathbf{K})\cong C_q(\mathcal{L}|_{V^Y},\theta;\mathbf{K})\quad\forall q=0,1,\cdots.$$

Actually, from the proof of [27, Cor.2.8] one can get the following stronger conclusion:

**Proposition 4.5.** *For any open neighborhood $U^Y$ of $\theta$ in $V^Y$ and the corresponding one of $\theta$ in $V^X$, $U^X=U^Y\cap X$, the inclusion*

$$\iota:\left(\mathcal{L}_0\cap U^X,\mathcal{L}_0\cap U^X\setminus\{\theta\}\right)\to\left(\mathcal{L}_0\cap U^Y,\mathcal{L}_0\cap U^Y\setminus\{\theta\}\right)$$

*induces isomorphisms*

$$\iota_*:H_*\left(\mathcal{L}_0\cap U^X,\mathcal{L}_0\cap U^X\setminus\{\theta\}\right)\to H_*\left(\mathcal{L}_0\cap U^Y,\mathcal{L}_0\cap U^Y\setminus\{\theta\}\right)$$

*for any Abel group $\mathbf{K}$, where $\mathcal{L}_0=\{x\in V\,|\,\mathcal{L}(x)\le 0\}$.*

*Proof.* By the excision property of the singular homology theory we only need to prove it for some open neighborhood $U^Y$ of $\theta$ in $V^Y$. By [27, Claim 1])

$$\|y\|_D=\|(P^0+P^-)y\|_Y+\|P^+y\|_Y$$

gives a norm on $Y$ equivalent to $\|\cdot\|_Y$. Let $\kappa_0\in(0,\kappa)$ be so small that

$$B^Y_{\kappa_0}:=\{y\in Y\,|\,\|y\|_D<\kappa_0\}\subset B_Y(\theta,\delta)\tag{4.1}$$

and that $U^Y=\Psi(B^Y_{\kappa_0})$ (resp. $\Psi(B^Y_{\kappa_0}\cap X)$) is a neighborhood of $\theta$ in $Y$ (resp. $X$) which only contains $\theta$ as a unique critical point of $\mathcal{L}|_{V^Y}$ (resp. $\mathcal{L}|_{V^X}$). (This can be assured by the second claim in Theorem 4.1). For conveniences let

$$\mathcal{Y}=\mathcal{L}_0\cap U^Y\quad\text{and}\quad\mathcal{X}=\mathcal{Y}\cap X=\mathcal{L}_0\cap U^X=\{y\in U^Y\cap X\,|\,\mathcal{L}(y)\le 0\},$$

and let $\iota : (\mathcal{X}, \mathcal{X} \setminus \{\theta\}) \hookrightarrow (\mathcal{Y}, \mathcal{Y} \setminus \{\theta\})$ be the inclusion. By Theorem 4.1 we have

$$\Psi^{-1}(\mathcal{Y}) = \left\{ y \in B^Y_{\kappa_0} \;\middle|\; \frac{1}{2}(B(\theta)y^\perp, y^\perp) + \mathcal{L}(z + \rho(z)) \le 0 \right\}$$

and

$$(\Psi^{-1}|_{\mathcal{Y}})_* : H_*(\mathcal{Y}, \mathcal{Y} \setminus \{\theta\}) \cong H_*(\Psi^{-1}(\mathcal{Y}), \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\}),$$
$$(\Psi^{-1}|_{\mathcal{X}})_* : H_*(\mathcal{X}, \mathcal{X} \setminus \{\theta\}) \cong H_*(\Psi^{-1}(\mathcal{Y}) \cap X, \Psi^{-1}(\mathcal{Y}) \cap X \setminus \{\theta\}).$$

Define $\Psi^{-1}(\mathcal{Y})_{0-} = \Psi^{-1}(\mathcal{Y}) \cap (H^0 + H^-)$. Then $\Psi^{-1}(\mathcal{Y})_{0-} \subset X$ and thus

$$\Psi^{-1}(\mathcal{Y})_{0-} = \Psi^{-1}(\mathcal{Y})_{0-} \cap X. \tag{4.2}$$

For $B^Y_{\kappa_0}$ in (4.1) let $\Re : [0,1] \times B^Y_{\kappa_0} \to Y$ be the continuous map defined by

$$\Re(t, y) = (P^0 + P^-)y + (1-t)P^+ y.$$

Clearly, $\Re(0, \cdot) = id$, $\Re(t, \cdot)|_{\Psi^{-1}(\mathcal{Y})_{0-}} = id$ and $\Re(1, \Psi^{-1}(\mathcal{Y})) \subset \Psi^{-1}(\mathcal{Y})_{0-}$. It was proved in [27] that $\Re$ is also a continuous map from $[0,1] \times (B^Y_{\kappa_0} \cap X)$ to $X$ (with respect to the induced topology from $X$) and that

(I) $\Re(1, \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\}) \subset \Psi^{-1}(\mathcal{Y})_{0-} \setminus \{\theta\}$,

(II) $\Re(t, \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\}) \subset \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\}$ for $t \in [0,1]$.

These show that $\Re$ gives not only a deformation retract from $(\Psi^{-1}(\mathcal{Y}), \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\})$ to $(\Psi^{-1}(\mathcal{Y})_{0-}, \Psi^{-1}(\mathcal{Y})_{0-} \setminus \{\theta\})$, but also one from $(\Psi^{-1}(\mathcal{Y}) \cap X, \Psi^{-1}(\mathcal{Y}) \cap X \setminus \{\theta\})$ to

$$(\Psi^{-1}(\mathcal{Y})_{0-} \cap X, \Psi^{-1}(\mathcal{Y})_{0-} \cap X \setminus \{\theta\}) = (\Psi^{-1}(\mathcal{Y})_{0-}, \Psi^{-1}(\mathcal{Y})_{0-} \setminus \{\theta\})$$

(with respect to the induced topology from $X$). Hence inclusions

$$i^y : (\Psi^{-1}(\mathcal{Y})_{0-}, \Psi^{-1}(\mathcal{Y})_{0-} \setminus \{\theta\}) \hookrightarrow (\Psi^{-1}(\mathcal{Y}), \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\}) \quad \text{and}$$
$$i^x : (\Psi^{-1}(\mathcal{Y})_{0-} \cap X, \Psi^{-1}(\mathcal{Y})_{0-} \cap X \setminus \{\theta\}) \hookrightarrow (\Psi^{-1}(\mathcal{Y}) \cap X, \Psi^{-1}(\mathcal{Y}) \cap X \setminus \{\theta\})$$

induce isomorphisms

$$H_*(\Psi^{-1}(\mathcal{Y})_{0-}, \Psi^{-1}(\mathcal{Y})_{0-} \setminus \{\theta\}) \xrightarrow{i^y_*} H_*(\Psi^{-1}(\mathcal{Y}), \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\}) \quad \text{and}$$
$$H_*(\Psi^{-1}(\mathcal{Y})_{0-} \cap X, \Psi^{-1}(\mathcal{Y})_{0-} \cap X \setminus \{\theta\}) \xrightarrow{i^x_*} H_*(\Psi^{-1}(\mathcal{Y}) \cap X, \Psi^{-1}(\mathcal{Y}) \cap X \setminus \{\theta\}).$$

Consider the inclusions

$$i^{xy} : (\Psi^{-1}(\mathcal{Y}) \cap X, \Psi^{-1}(\mathcal{Y}) \cap X \setminus \{\theta\}) \hookrightarrow (\Psi^{-1}(\mathcal{Y}), \Psi^{-1}(\mathcal{Y}) \setminus \{\theta\}) \quad \text{and}$$
$$i^{xy}_0 : (\Psi^{-1}(\mathcal{Y})_{0-} \cap X, \Psi^{-1}(\mathcal{Y})_{0-} \cap X \setminus \{\theta\}) \hookrightarrow (\Psi^{-1}(\mathcal{Y})_{0-}, \Psi^{-1}(\mathcal{Y})_{0-} \setminus \{\theta\}).$$

It is obvious that $i^{xy}\circ i^x = i^y\circ i_0^{xy}$. Since $H^0+H^-\subset X$, both $(H^0+H^-,\|\cdot\|_X)$ and $(H^0+H^-,\|\cdot\|_Y)$ are complete. Hence the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent on $H^0+H^-$. It follows from this and (4.2) that $i_0^{xy}$ is a homeomorphism. This shows that $(i_0^{xy})_*$ and hence $i_*^{xy}$ is an isomorphism. Note that $(\Psi^{-1}|_{\mathcal{Y}})\circ\iota = i^{xy}\circ(\Psi^{-1}|_{\mathcal{X}})$. Proposition 4.5 follows immediately. $\square$

Before proving Theorem 2.10 we also need the following observation, which is contained in the proof of [11, Th.3.2, page 100] and seems to be obvious. But the author cannot find where it is explicitly pointed out.

**Remark 4.6.** Let $H$ be a real Hilbert space, and let $f\in C^2(H,\mathbb{R})$ satisfy the (PS) condition. Assume that $df(x)=x-Tx$, where $T$ is a compact mapping, and that $p_0$ is an isolated critical point of $f$. Then for any field $\mathbb{F}$ and each $q\in\mathbb{N}\cup\{0\}$, $C_q(f,p_0;\mathbb{F})$ is a finite dimension vector space over $\mathbb{F}$. In particular, if $f\in C^2(\mathbb{R}^n,\mathbb{R})$ has an isolated critical point $p_0\in\mathbb{R}^n$ then $C_q(f,p_0;\mathbb{F})$, $q=0,1,\cdots$, are vector spaces over $\mathbb{F}$ of finite dimensions. In fact, by [11, (3.2), page 101] we have

$$C_*(f,p_0;\mathbb{F}) = H_*(W,W_-;\mathbb{F}) = H_*\left(\tilde{f}_{\frac{2}{3}\gamma}\cap W, \tilde{f}_{-\frac{2}{3}\gamma}\cap W;\mathbb{F}\right),$$

where $(W,W_-)$ is a Gromoll-Meyer pair of $f$ at $p_0$, and $\tilde{f}$ has only nondegenerate critical points $\{p_j\}_1^m$ in $W$, finite in number, contained in $B_H(p_0,\delta)\subset\mathrm{Int}(W)\cap f^{-1}[-\gamma/3,\gamma/3]$. Hence $C_*(f,p_0;\mathbb{F}) = \oplus_{j=1}^m C_*(\tilde{f},p_j;\mathbb{F})$. The claim follows because each $C_q(\tilde{f},p_j;\mathbb{F})$ is either $\mathbb{F}$ or $0$.

*Proof of Theorem 2.10.* By assumptions $(X,H,\mathcal{L})$ and $(X,Y,H,\mathcal{L})$ satisfy the conditions in Corollary 2.6 and in Corollaries 4.3, 4.4 respectively. By Remark 4.2 near $\theta\in H^0$ the maps $h$ and $\rho$ are same. Then Corollaries 2.6, 4.3 and 4.4 lead to

$$C_*(\mathcal{L},\theta;\mathbf{K})\cong C_*(\mathcal{L}|_{V^Y},\theta;\mathbf{K})\cong C_*(\mathcal{L}|_{V^X},\theta;\mathbf{K}) \tag{4.3}$$

for any Abel group $\mathbf{K}$.

Note that we may assume that $W$ is given by Theorem 2.1 because of the excision property of the singular homology groups. By Proposition 4.5 the inclusion

$$I^{xy}:\left(\mathcal{L}_0\cap W^X,\mathcal{L}_0\cap W^X\setminus\{\theta\}\right)\hookrightarrow\left(\mathcal{L}_0\cap W^Y,\mathcal{L}_0\cap W^Y\setminus\{\theta\}\right)$$

induces isomorphisms $I_*^{xy}$,

$$H_*\left(\mathcal{L}_0\cap W^X,\mathcal{L}_0\cap W^X\setminus\{\theta\};\mathbf{K}\right)\to H_*\left(\mathcal{L}_0\cap W^Y,\mathcal{L}_0\cap W^Y\setminus\{\theta\};\mathbf{K}\right).$$

By (4.3) and Remark 4.6, for a field $\mathbb{F}$ and each $q\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}
&C_q(\mathcal{L}|_{V^X},\theta;\mathbb{F})\cong H_q\left(\mathcal{L}_0\cap W^X,\mathcal{L}_0\cap W^X\setminus\{\theta\};\mathbb{F}\right),\\
&C_q(\mathcal{L}|_{V^Y},\theta;\mathbb{F})\cong H_q\left(\mathcal{L}_0\cap W^Y,\mathcal{L}_0\cap W^Y\setminus\{\theta\};\mathbb{F}\right),\\
&C_q(\mathcal{L},\theta;\mathbb{F})\cong H_q\left(\mathcal{L}_0\cap W,\mathcal{L}_0\cap W\setminus\{\theta\};\mathbb{F}\right)
\end{aligned}$$

are isomorphic vector spaces over $\mathbb{F}$ of finite dimension. Then any surjective (or injective) homomorphism among them must be an isomorphism. By Corollary 2.5 $I_*^{xw}$ is a surjection and hence an isomorphism. Since $I_*^{xw} = I_*^{yw} \circ I_*^{xy}$, $I_*^{yw}$ is also an isomorphism. $\square$

# 5 Proof of Theorem 2.12

We use the ideas of [24] to prove (i) in Step 1, and then derive (ii) in Step 2 from [14, Th.1.2] by checking that $\nabla\mathcal{L}$ is a demicontinuous map of class $(S)_+$.

**Step 1**. By the first paragraph in Step 1 of Lemma 3.1, $(I - P^0)B(\theta)|_{X^\pm} : X^\pm \to X^\pm$ is a Banach isomorphism. Consider the $C^1$ map $\Theta : [2,3] \times (V \cap X^\pm) \to X^\pm$ given by

$$(t, u) \mapsto (3 - t)(I - P^0)A(u) + (t - 2)(I - P^0)B(\theta)u. \tag{5.1}$$

Then $D_2\Theta(t,\theta) = (I - P^0)B(\theta)|_{X^\pm}$ for all $t \in [2,3]$. By the inverse function theorem there exist positive constants $\rho \in (0, r_0]$ and $C_7 > 0, C_8 > 0$ such that

$$C_7\|u\|_X \le \|\Theta(t,u)\|_X \le C_8\|u\|_X \quad \forall u \in B_X(\theta,\rho) \cap X^\pm,\ t \in [2,3]. \tag{5.2}$$

Following the notations in Lemma 3.1, we can shrink $\rho > 0$ (if necessary) such that the following (i)-(iii) are satisfied:

**(i)** $\theta$ is a unique zero of $A$ in $B_X(\theta, 2\rho)$,

**(ii)** $z + h(z) \in B_X(\theta, r_0/2)$ for any $z \in B_X(\theta, 2\rho)$,

**(iii)** $\|z\|_X < r_0$ and $\|u\|_X < r_0$ for any $z + u \in B_X(\theta, 2\rho)$. (This is possible because $z$ belongs to the finite dimension space $H^0$.)

Now we define a map $\Gamma : [0,3] \times B_X(\theta,\rho) \to X,\ (t, z + u) \mapsto \Gamma_t(z + u)$, where

$$\Gamma_t(z+u) = \begin{cases} (I - P^0)A(z+u) + P^0A\big(th(z) + (1-t)u + z\big) & \text{if } t \in [0,1], \\ (I - P^0)A\big(u + (2-t)z\big) + P^0A(z + h(z)) & \text{if } t \in [1,2], \\ (3-t)(I-P^0)A(u) + (t-2)(I-P^0)A'(\theta)u + P^0A(z+h(z)) & \\ \qquad\qquad \text{if } t \in [2,3]. \end{cases}$$

Clearly, $\Gamma$ is $C^0$, and every $\Gamma_t$ is $C^1$ and satisfies $\Gamma_t(\theta) = \theta$. Let us prove:

**Claim 5.1**. $\exists\ \epsilon \in (0,\rho)$ such that $\Gamma_t(x) \ne \theta\ \forall (t,x) \in [0,3] \times (\bar{B}_X(\theta,\epsilon) \setminus \{\theta\})$.

In fact, assume that $\Gamma_t(z+u) = \theta$ for some $t \in [0,1]$ and $z + u \in \bar{B}_X(\theta,\rho)$. Then $(I - P^0)A(z+u) + P^0A\big(th(z) + (1-t)u + z\big) = \theta$ and hence

$$(I - P^0)A(z+u) = \theta \quad \text{and} \quad P^0A\big(th(z) + (1-t)u + z\big) = \theta.$$

By the first equality, (3.5) and the uniqueness we have $u = h(z)$. So the second equality becomes

$$\theta = P^0A\big(th(z) + (1-t)u + z\big) = P^0A\big(th(z) + (1-t)h(z) + z\big) = P^0A\big(z + h(z)\big).$$

This and (3.5) give $A(z+h(z))=\theta$. By (i) we get $z+h(z)=\theta$. That is, $z=\theta$ and $z+u=\theta$.

Similarly, let $\Gamma_t(z+u)=\theta$ for some $t\in[1,2]$ and $z+u\in\bar{B}_X(\theta,\rho)$. Then

$$(I-P^0)A\big(u+(2-t)z\big)=\theta \quad \text{and} \quad P^0A(z+h(z))=\theta.$$

(3.5) and the second equality yield $A(z+h(z))=\theta$, and hence $z=\theta$ as above. Since $\|u\|_X<r_0<r_1$, it follows from the first equality and the construction of $h$ above (3.5) that $u=h((2-t)z)=\theta$.

Finally, assume that $\Gamma_t(z+u)=\theta$ for some $t\in[2,3]$ and $z+u\in B_X(\theta,\epsilon)$, where $\epsilon\in(0,\rho)$ is such that $\|u\|_X<\rho$ for any $z+u\in B_X(\theta,\epsilon)$ (with $z\in H^0$ and $u\in X^{\pm}$). Then $P^0A(z+h(z))=\theta$ and

$$\Theta(t,u)=(3-t)(I-P^0)A(u)+(t-2)(I-P^0)B(\theta)u=\theta.$$

The former implies $z=\theta$ as above, and (5.2) leads to $u=\theta$. Claim 5.1 is proved.

By Lemma 3.1(i), $h'(\theta)=\theta$. Using this it is easily proved that $d\Gamma_t(\theta)=A'(\theta)$ for any $t\in[0,3]$. Since the $C^1$ Fredholm map is locally proper, we can shrink $\epsilon>0$ such that the restriction of each $\Gamma_t$ to $\bar{B}_X(\theta,\epsilon)$ is Fredholm and that the restriction of $\Gamma$ to $[0,3]\times\bar{B}_X(\theta,\epsilon)$ is proper. Hence $\Gamma:[0,3]\times B_X(\theta,\epsilon)\to X$ satisfies the homotopy definition in the Benevieri-Furi degree theory [5, 6], and we arrive at

$$\deg_{\rm BF}(A,B_X(\theta,\epsilon),\theta)=\deg_{\rm BF}(\Gamma_0,B_X(\theta,\epsilon),\theta)=\deg_{\rm BF}(\Gamma_3,B_X(\theta,\epsilon),\theta). \tag{5.3}$$

Recall that $D\Gamma_3(\theta)=A'(\theta)=B(\theta)|_X$ and

$$\Gamma_3(z+u)=(I-P^0)A'(\theta)u+P^0A(z+h(z))=I-[P^0B(\theta)u-P^0A(z+h(z))].$$

Moreover $\dim H^0<\infty$ implies that the map

$$\bar{B}_X(\theta,\epsilon)\to X,\ z+u\mapsto K(z+u):=P^0B(\theta)u-P^0A(z+h(z))$$

is compact. Hence the Leray-Schauder degree $\deg_{\rm LS}(I-K,B_X(\theta,\epsilon),\theta)$ exists, and

$$\begin{aligned}\deg_{\rm FPR}(I-K,B_X(\theta,\epsilon),\theta) &= \deg_{\rm BF}(I-K,B_X(\theta,\epsilon),\theta)\\ &= \deg_{\rm LS}(I-K,B_X(\theta,\epsilon),\theta)\end{aligned} \tag{5.4}$$

for a suitable orientation of the map $I-K$. By Remark 3.2 and Lemma 3.1 $\mathcal{L}^\circ$ is $C^2$ and

$$d\mathcal{L}^\circ(z_0)(z)=(A(z_0+h(z_0)),z)_H \quad \forall z_0\in B_{H^0}(\theta,r_0),\ z\in H^0.$$

Hence the gradient of $\mathcal{L}^\circ$ with respect to the induced inner on $H^0$ (from $H$), denoted by $\nabla\mathcal{L}^\circ$, is given by $\nabla\mathcal{L}^\circ(z)=P^0A(z+h(z))\ \forall z\in B_{H^0}(\theta,r_0)$. By the definition and properties of the Leray-Schauder degree it is easily proved that

$$\deg_{\rm LS}(I-K,B_X(\theta,\epsilon),\theta)=(-1)^{\dim H^-}\deg_{\rm LS}(\nabla\mathcal{L}^\circ,B_{H^0}(\theta,\epsilon),\theta) \tag{5.5}$$

Moreover, $B_X(\theta,\epsilon)$ is open, connected and simply connected. After a suitable orientation is chosen it follows from (5.3)-(5.5) that

$$\begin{aligned}\deg_{\rm FPR}(A, B_X(\theta,\epsilon),\theta) &= \deg_{\rm BF}(A, B_X(\theta,\epsilon),\theta)\\ &= (-1)^{\dim H^-}\deg_{\rm LS}(\nabla\mathcal{L}^\circ, B_X(\theta,\epsilon)\cap H^0,\theta)\\ &= (-1)^{\dim H^-}\sum_{q=0}^{\infty}(-1)^q{\rm rank}C_q(\mathcal{L}^\circ,\theta;\mathbf{K}),\end{aligned}$$

where the final equality comes from [37, Th.8.5]. Combing this with Corollary 2.6 the expected first conclusion is obtained.

**Step 2**. Recall that a map $T$ from a reflexive real Banach space to its dual $X^*$ is said to be *demicontinuous* if $T$ maps strongly convergent sequences in $X$ to weakly convergent sequences in $X^*$. Now since the Hilbert space $H$ is self-adjoint and $D\mathcal{L}(x)(u) = (\nabla\mathcal{L}(x),u)_H$, by the continuously directional differentiability of $\mathcal{L}$, if $\{x_n\}\subset V$ converges to $x\in V$ in $H$ then $\{\nabla\mathcal{L}(x_n)\}$ weakly converges to $\nabla\mathcal{L}(x)$, i.e., $(\nabla\mathcal{L}(x_n),u)_H\to(\nabla\mathcal{L}(x),u)_H$ for every $u\in H$. This shows that the map $\nabla\mathcal{L}: V\to H=H^*$ is demicontinuous in the sense of [9, Th.4].

Next we show that the restriction of $\nabla\mathcal{L}$ to a small neighborhood of $\theta\in H$ is of class $(S)_+$ in the sense of [9, Def.2(b)]. By (D3), for the constants $\eta_0$ and $C_0'$ in (D4*) and $\rho>0$ in (i)-(iii) above we can choose $\rho_0\in(0,\rho)$ such that $2\rho_0<\eta_0$ and the following (iv)-(v) are satisfied:

**(iv)** $B_{H^0}(\theta,2\rho_0)\subset B_X(\theta,\rho)$ and

$$\|Q(x)-Q(\theta)\|<\frac{C_0'}{2}\quad\forall x\in B_H(\theta,2\rho_0)\cap X;\tag{5.6}$$

**(v)** $\theta$ is a unique zero of $\nabla\mathcal{L}$ in $B_H(\theta,2\rho_0)\subset V$.

Then (5.6) and (D4*) yield

$$\begin{aligned}\big(B(x)u,u\big)_H &= \big(P(x)u,u\big)_H+\big([Q(x)-Q(\theta)]u,u\big)_H+\big(Q(\theta)u,u\big)_H\\ &\ge \frac{C_0'}{2}\|u\|^2+\big(Q(\theta)u,u\big)_H\end{aligned}\tag{5.7}$$

for all $x\in B_H(\theta,2\rho_0)\cap X$ and $u\in H$. Take $\rho_1\in(0,\rho_0)$ so small that

$$z+h(z)\in B_H(\theta,\rho_0)\quad\forall z\in B_{H^0}(\theta,2\rho_1).$$

(This assures that the functional $\mathcal{L}^\circ$ in Corollary 2.6 is defined on $B_{H^0}(\theta,2\rho_1)$). Then for $x,x'\in B_H(\theta,2\rho_1)\cap X$, by (F2)-(F3) and the mean value theorem we have $\tau\in(0,1)$

such that

$$\begin{aligned}
& (\nabla\mathcal{L}(x), x - x')_H \\
= \;& (\nabla\mathcal{L}(x) - \nabla\mathcal{L}(x'), x - x')_H - (\nabla\mathcal{L}(x'), x - x')_H \\
= \;& (A(x) - A(x'), x - x')_H - (\nabla\mathcal{L}(x'), x - x')_H \\
= \;& \big(DA([\tau x + (1-\tau)x'])(x - x'), x - x'\big)_H - (\nabla\mathcal{L}(x'), x - x')_H \\
= \;& \big(B([\tau x + (1-\tau)x'])(x - x'), x - x'\big)_H - (\nabla\mathcal{L}(x'), x - x')_H \\
\geq \;& \frac{C_0'}{2}\|x - x'\|^2 - (\nabla\mathcal{L}(x'), x - x')_H + (Q(\theta)(x - x'), x - x')_H,
\end{aligned}$$

where the final inequality is because of (5.7). Since $\mathcal{L}$ is continuously directional differentiable and $B_H(\theta, 2\rho_1) \cap X$ is dense in $B_H(\theta, 2\rho_1)$ we obtain

$$\begin{aligned}
(\nabla\mathcal{L}(x), x - x')_H \;\geq\; & \frac{C_0'}{2}\|x - x'\|^2 - (\nabla\mathcal{L}(x'), x - x')_H \\
+ \; & (Q(\theta)(x - x'), x - x')_H
\end{aligned} \tag{5.8}$$

for any $x, x' \in B_H(\theta, 2\rho_1)$.

Let $\{x_n\} \subset B_H(\theta, 2\rho_1)$ weakly converge to $x \in B_H(\theta, 2\rho_1)$ and

$$\overline{\lim_{n\to\infty}}(\nabla\mathcal{L}(x_n), x_n - x)_H \leq 0.$$

Then $(\nabla\mathcal{L}(x), x_n - x)_H \to 0$, and $(Q(\theta)(x_n - x), x_n - x)_H \to 0$ by the compactness of $Q(\theta)$. It follows from these and (5.8) that

$$\frac{C_0'}{2}\lim_{n\to\infty}\|x_n - x\| \leq \frac{C_0'}{2}\overline{\lim_{n\to\infty}}\|x_n - x\|^2 \leq \overline{\lim_{n\to\infty}}(\nabla\mathcal{L}(x_n), x_n - x)_H \leq 0,$$

This is, $\lim_{n\to\infty}\|x_n - x\| = 0$. Hence the map $\nabla\mathcal{L} : B_H(\theta, 2\rho_1) \to H$ is of class $(S)_+$.

Then three equalities in the formula of Theorem 2.12(ii) follow from [14, Th.1.2], Corollary 2.6 and [37, Th.8.5], respectively. □

# 6 The functor properties of the splitting lemma

The splitting lemma for $C^2$ functionals on Hilbert spaces has some natural functor properties. This section studies some corresponding properties in our setting.

Consider a tuple $(H, X, \mathcal{L}, A, B = P + Q)$, where $H$ (resp. $X$) is a Hilbert (resp. Banach) space satisfying the condition (S) as in Section 2, the functional $\mathcal{L} : H \to \mathbb{R}$ and maps $A : X \to X$ and $B : X \to L_s(H)$ satisfy, at least near the origin $\theta \in H$, the conditions (F1)-(F3), (C1)-(C2) and (D) in Section 2. (We can assume that these conditions are satisfied on $H$ without loss of generality.)

Let $(\widehat{H}, \widehat{X}, \widehat{\mathcal{L}}, \widehat{A}, \widehat{B} = \widehat{P} + \widehat{Q})$ be another such a tuple. Suppose that $J : H \to \widehat{H}$ is a linear injection satisfying:

$$(Ju, Jv)_{\widehat{H}} = (u, v)_H \quad \forall u, v \in H, \tag{6.1}$$

$$J(X) \subset \widehat{X} \quad \text{and} \quad J|_X \in L(X, \widehat{X}). \tag{6.2}$$

Furthermore, we assume

$$\widehat{\mathcal{L}} \circ J = \mathcal{L}, \tag{6.3}$$

which implies

$$\widehat{A}(J(x)) = J \circ A(x) \quad \forall x \in X, \tag{6.4}$$

$$\widehat{B}(J(x)) \circ J = J \circ B(x)\ \forall x \in X. \tag{6.5}$$

Let $H = H^0 \oplus H^+ \oplus H^-$, $X = H^0 \oplus X^+ \oplus X^-$ and $\widehat{H} = \widehat{H}^0 \oplus \widehat{H}^+ \oplus \widehat{H}^-$ and $\widehat{X} = \widehat{H}^0 \oplus \widehat{X}^+ \oplus \widehat{X}^-$ be the corresponding decompositions. Namely, $\widehat{H}^0 = \mathrm{Ker}(\widehat{B}(\theta))$, and $\widehat{H}^+$ (resp. $\widehat{H}^-$) is the positive (resp. negative) definite subspace of $\widehat{B}(\theta)$. Denote by $P^*$ (resp. $\widehat{P}^*$) the orthogonal projections from $H$ (resp. $\widehat{H}$) to $H^*$ (resp. $\widehat{H}^*$) for $* = +, -, 0$. Since $\widehat{B}(\theta) \circ J = J \circ B(\theta)$ by (6.5), we have

$$JH^\star \subset \widehat{H}^\star, \quad \widehat{P}^\star \circ J = J \circ P^\star,\ \star = -, 0, +. \tag{6.6}$$

**Claim 6.1**. $(\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1} \circ (J|_{X^\pm}) = J|_{X^\pm} \circ (B(\theta)|_{X^\pm})^{-1}$.

In fact, for $v \in X^\pm$ let $y = (\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1} \circ (J|_{X^\pm})v$. Then $y \in \widehat{X}^\pm$ because $J(X^\pm) \subset \widehat{X}^\pm$ by (6.2) and (6.6), and $Jv = \widehat{B}(\theta)y$. Note that we may write $v = B(\theta)|_{X^\pm}u$ for a unique $u \in X^\pm$. It follows that $J|_{X^\pm} \circ B(\theta)|_{X^\pm}u = \widehat{B}(\theta)|_{\widehat{X}^\pm}y$ and hence $\widehat{B}(\theta)(Ju) = \widehat{B}(\theta)y$ by (6.5). The latter implies $Ju = y$ since both $Ju$ and $y$ sit in $\widehat{X}^\pm$. From this and (6.5) we deduce that $Jv = \widehat{B}(\theta)y = \widehat{B}(\theta)(Ju) = J \circ B(\theta)u$ and hence $v = B(\theta)u$. Then $(\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1} \circ (J|_{X^\pm})v = y = Ju = J \circ (B(\theta)|_{X^\pm})^{-1}v$. Claim 6.1 is proved.

Assume that the nullity of $\mathcal{L}$ at $\theta \in H$

$$\nu(\mathcal{L}, \theta) := \dim H^0 > 0 \quad \text{and hence} \quad \nu(\widehat{\mathcal{L}}, \theta) > 0 \tag{6.7}$$

by (6.6). Here $\nu(\widehat{\mathcal{L}}, \theta) := \dim \widehat{H}^0$ is nullity of $\widehat{\mathcal{L}}$ at $\theta \in \widehat{H}$. Corresponding to the map $S$ in (3.3) let us consider the map

$$\widehat{S} : B_{\widehat{H}^0}(\theta, r_1) \times (B_{\widehat{X}}(\theta, r_1) \cap \widehat{X}^\pm) \to \widehat{X}^\pm,$$
$$\widehat{S}(\hat{z}, \hat{x}) = -(\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1}(I_{\widehat{X}} - \widehat{P}^0)\widehat{A}(\hat{z} + \hat{x}) + \hat{x}$$

for $\hat{z}_1, \hat{z}_2 \in B_{\widehat{H}^0}(\theta, r_1)$ and $\hat{x}_1, \hat{x}_2 \in B_{\widehat{X}}(\theta, r_1) \cap \widehat{X}^\pm$. (Here $\widehat{X}^\pm = \widehat{X}^+ \oplus \widehat{X}^-$, and we may shrink $r_1 > 0$ if necessary). Then for all $z \in B_{H^0}(\theta, r_1)$ and $x \in B_X(\theta, r_1) \cap X^\pm$ we

derive from (6.4) and Claim 6.1 that

$$\begin{aligned}\widehat{S}(Jz, Jx) &= -(\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1}(I_{\widehat{X}} - \widehat{P}^0)\widehat{A}(Jz + Jx) + Jx\\ &= -(\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1}(I_{\widehat{X}} - \widehat{P}^0)\circ J\circ A(z + x) + Jx\\ &= -(\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1}\circ J\circ(I_X - P^0)A(z + x) + Jx\\ &= -J\circ(B(\theta)|_{X^\pm})^{-1}\circ(I_X - P^0)A(z + x) + Jx.\end{aligned}$$

That is, for all $z \in B_{H^0}(\theta, r_1)$ and $x \in B_X(\theta, r_1) \cap X^\pm$ it holds that

$$\widehat{S}(Jz, Jx) = J \circ S(z, x). \tag{6.8}$$

From the proof of Lemma 3.1 there exist $r_0 \in (0, r_1)$ and a unique map $\hat{h} : B_{\widehat{H}^0}(\theta, r_1) \to B_{\widehat{X}}(\theta, r_1) \cap \widehat{X}^\pm$ such that $\hat{h}(\theta) = \theta$ and

$$\widehat{S}(\hat{z}, \hat{h}(\hat{z})) = \hat{h}(\hat{z}) \quad \text{(or equivalently } (I_{\widehat{X}} - \widehat{P}^0)\widehat{A}(\hat{z} + \hat{h}(\hat{z})) = 0).$$

Moreover, $\hat{h}$ satisfies the corresponding conclusions in Lemma 3.1. For $z \in B_{H^0}(\theta, r_0)$ we have also $(I_X - P^0)A(z+h(z)) = 0$, i.e., $S(z, h(z)) = h(z)$. Hence by the uniqueness and (6.8) we arrive at

$$\hat{h}(Jz) = J \circ h(z) \quad \forall z \in B_{H^0}(\theta, r_0). \tag{6.9}$$

As in (3.12), we have a map $\widehat{F} : \bar{B}_{\widehat{H}^0}(\theta, \delta) \times B_{\widehat{H}^\pm}(\theta, \delta) \to \mathbb{R}$ given by

$$\widehat{F}(\hat{z}, \hat{u}) = \widehat{\mathcal{L}}(\hat{z} + \hat{h}(\hat{z}) + \hat{u}) - \widehat{\mathcal{L}}(\hat{z} + \hat{h}(\hat{z})). \tag{6.10}$$

Clearly, (6.3), (6.9) and (6.10) lead to

$$\widehat{F}(Jz, Ju) = F(z, u) \quad \forall (z, u) \in \bar{B}_{H^0}(\theta, \delta) \times B_{H^\pm}(\theta, \delta). \tag{6.11}$$

By shrinking $\varepsilon > 0$ in Lemma 3.5 (if necessary) we may assume that the restriction of $\widehat{F}$ to $\bar{B}_{\widehat{H}^0}(\theta, \varepsilon) \times \big(\bar{B}_{\widehat{H}^+}(\theta, \varepsilon) \oplus \bar{B}_{\widehat{H}^-}(\theta, \varepsilon)\big)$ satisfies the conditions in Theorem A.1. Then we have a homoeomorphism as in (3.29),

$$\begin{gathered}\widehat{\Phi} : B_{\widehat{H}^0}(\theta, \varepsilon) \times \big(B_{\widehat{H}^+}(\theta, \epsilon) + B_{\widehat{H}^-}(\theta, \epsilon)\big) \to \widehat{H},\\ (\hat{z}, \hat{u}^+ + \hat{u}^-) \mapsto \hat{z} + \hat{h}(\hat{z}) + \widehat{\phi}_{\hat{z}}(\hat{u}^+ + \hat{u}^-),\end{gathered} \tag{6.12}$$

such that $\widehat{\phi}_{\hat{z}}(\theta) = \theta$ and

$$\widehat{\mathcal{L}}(\widehat{\Phi}(\hat{z}, \hat{u}^+, \hat{u}^-)) = \widehat{\mathcal{L}}(\hat{z} + \hat{h}(\hat{z})) + (\hat{u}^+, \hat{u}^+)_{\widehat{H}} - (\hat{u}^-, \hat{u}^-)_{\widehat{H}}$$

for all $(\hat{z}, \hat{u}^+, \hat{u}^-) \in \bar{B}_{\widehat{H}^0}(\theta, \varepsilon) \times B_{\widehat{H}^+}(\theta, \epsilon) \times B_{\widehat{H}^-}(\theta, \epsilon)$.

**Claim 6.2.** *Under the assumptions above, if*

$$\mu(\mathcal{L}, \theta) = \mu(\widehat{\mathcal{L}}, \theta), \tag{6.13}$$

*then* $\widehat{\Phi}(Jz, Ju^+ + Ju^-) = J \circ \Phi(z, u^+ + u^-)$ *for* $(z, u^+, u^-) \in \bar{B}_{H^0}(\theta, \varepsilon) \times B_{H^+}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon)$. *Here* $\mu(\mathcal{L}, \theta) := \dim H^-$ *and* $\mu(\widehat{\mathcal{L}}, \theta) := \dim \widehat{H}^-$.

In fact, suppose $\mu(\mathcal{L}, \theta) = \mu(\widehat{\mathcal{L}}, \theta) = 0$. By $1^\circ$) in the proof of Theorem A.1

$$\widehat{\psi}(\hat{z}, \hat{x}) = \begin{cases} \frac{\sqrt{\widehat{\mathcal{L}}(\hat{z}+\hat{h}(\hat{z})+\hat{x})-\widehat{\mathcal{L}}(\hat{z}+\hat{h}(\hat{z}))}}{\|\hat{x}\|_{\widehat{H}}}\hat{x} & \text{if } \hat{x} \neq \theta, \\ \theta & \text{if } \hat{x} = \theta \end{cases}$$

for all $(\hat{z}, \hat{x}) \in \bar{B}_{\widehat{H}^0}(\theta, \varepsilon) \times B_{\widehat{H}^\pm}(\theta, \epsilon_1)$, and

$$\psi(z, x) = \begin{cases} \frac{\sqrt{\mathcal{L}(z+h(z)+x)-\mathcal{L}(z+h(z))}}{\|x\|_H}x & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta \end{cases}$$

for all $(z, x) \in \bar{B}_{H^0}(\theta, \varepsilon) \times B_{H^\pm}(\theta, \epsilon_1)$. It follows from (6.3) and (6.9) that

$$\widehat{\psi}(Jz, Ju) = J \circ \psi(z, u) \quad \text{and thus} \quad \widehat{\phi}_{Jz}(Ju) = J \circ \phi_z(u)$$

for $(z, u) \in \bar{B}_{H^0}(\theta, \varepsilon) \times B_{H^\pm}(\theta, \epsilon)$. The desired results follow from (3.29) and (6.12).

Next suppose $\mu(\mathcal{L}, \theta) = \mu(\widehat{\mathcal{L}}, \theta) > 0$. Recall the constructions of $\phi_z$ and $\widehat{\phi}_{\hat{z}}$. By (A.9),

$$\widehat{\phi}_{\hat{z}}(\hat{u}^+ + \hat{u}^-) = \hat{x}^+ + \hat{x}^-$$

for any $(\hat{z}, \hat{u}^+, \hat{u}^-) \in \bar{B}_{\widehat{H}^0}(\theta, \varepsilon) \times B_{\widehat{H}^+}(\theta, \epsilon) \times B_{\widehat{H}^-}(\theta, \epsilon)$, where $(\hat{x}^+, \hat{x}^-)$ is a unique point in $B_{\widehat{H}^+}(\theta, 2\epsilon) \times B_{\widehat{H}^-}(\theta, \delta)$ satisfying $\widehat{\psi}(\hat{z}, \hat{x}^+ + \hat{x}^-) = \hat{u}^+ + \hat{u}^-$. By Step 4 in the proof of Theorem A.1 we know

$$\widehat{\psi}(\hat{z}, \hat{x}^+ + \hat{x}^-) = \widehat{\psi}_1(\hat{z}, \hat{x}^+ + \hat{x}^-) + \widehat{\psi}_2(\hat{z}, \hat{x}^+ + \hat{x}^-)$$

for all $(\hat{z}, \hat{x}^+, \hat{x}^-) \in \bar{B}_{\widehat{H}^0}(\theta, \varepsilon) \times B_{\widehat{H}^+}(\theta, \epsilon_1) \times B_{\widehat{H}^-}(\theta, \delta)$, where

$$\widehat{\psi}_1(\hat{z}, \hat{x}^+ + \hat{x}^-) = \begin{cases} \frac{\sqrt{\widehat{F}(\hat{z}, \hat{x}^+ + \widehat{\varphi}_{\hat{z}}(\hat{x}^+))}}{\|\hat{x}^+\|_{\widehat{H}}}\hat{x}^+ & \text{if } \hat{x}^+ \neq \theta, \\ \theta & \text{if } \hat{x}^+ = \theta \end{cases}$$

and

$$\widehat{\psi}_2(\hat{z}, \hat{x}^+ + \hat{x}^-) = \begin{cases} \frac{\sqrt{\widehat{F}(\hat{z}, \hat{x}^+ + \widehat{\varphi}_{\hat{z}}(\hat{x}^+)) - \widehat{F}(\hat{z}, \hat{x}^+ + \hat{x}^-)}}{\|\hat{x}^- - \widehat{\varphi}_{\hat{z}}(\hat{x}^+)\|_{\widehat{H}}}(\hat{x}^- - \widehat{\varphi}_{\hat{z}}(\hat{x}^+)) & \text{if } \hat{x}^- \neq \widehat{\varphi}_{\hat{z}}(\hat{x}^+), \\ \theta & \text{if } \hat{x}^- = \widehat{\varphi}_{\hat{z}}(\hat{x}^+). \end{cases}$$

Here for each $(\hat{z}, \hat{x}^+) \in \bar{B}_{\widehat{H}^0}(\theta, \varepsilon) \times B_{\widehat{H}^+}(\theta, \epsilon_1)$, as showed in Step 1 of the proof of Theorem A.1, $\widehat{\varphi}_{\hat{z}}(\hat{x}^+)$ is a unique point in $B_{\widehat{H}^-}(\theta, \delta)$ such that

$$\widehat{F}(\hat{z}, \hat{x}^+ + \widehat{\varphi}_{\hat{z}}(\hat{x}^+)) = \max\big\{\widehat{F}(\hat{z}, \hat{x}^+ + \hat{x}^-) \,|\, \hat{x}^- \in B_{\widehat{H}^-}(\theta, \delta)\big\}.$$

For $(z,x^+)\in \bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^+}(\theta,\epsilon_1)$ we have $(Jz,Jx^+)\in \bar{B}_{\widehat{H}^0}(\theta,\varepsilon)\times B_{\widehat{H}^+}(\theta,\epsilon_1)$ by (6.6), and $J(B_{H^-}(\theta,\delta))=B_{\widehat{H}^-}(\theta,\delta)$ by (6.1), (6.6) and (6.13). These and (6.11) lead to

$$\begin{aligned}\widehat{F}(Jz,Jx^++\widehat{\varphi}_{Jz}(Jx^+)) &= \max\big\{\widehat{F}(Jz,Jx^++\hat{x}^-)\,|\,\hat{x}^-\in B_{\widehat{H}^-}(\theta,\delta)\big\}\\ &= \max\big\{\widehat{F}(Jz,Jx^++\hat{x}^-)\,|\,\hat{x}^-\in J(B_{H^-}(\theta,\delta))\big\}\\ &= \max\big\{F(z,x^++x^-)\,|\,x^-\in B_{H^-}(\theta,\delta)\big\}\\ &= F(z,x^++\varphi_z(x^+))\\ &= \widehat{F}(Jz,Jx^++J\varphi_z(x^+)).\end{aligned}$$

By the uniqueness we arrive at

$$\widehat{\varphi}_{Jz}(Jx^+)=J\varphi_z(x^+)\quad \forall(z,x^+)\in\bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^+}(\theta,\epsilon_1),$$

which implies

$$\widehat{\psi}(Jz,Jx^++Jx^-)=J\circ\psi(z,x^++x^-)$$

for all $(z,x^+,x^-)\in\bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^+}(\theta,\epsilon_1)\times B_{H^-}(\theta,\delta)$. From (6.6) and the definition of $\widehat{\phi}_{\hat{z}}(\hat{u}^++\hat{u}^-)$ we deduce that

$$\widehat{\phi}_{Jz}(Ju^++Ju^-)=J\circ\phi_z(u^++u^-)$$

for $(z,u^+,u^-)\in\bar{B}_{H^0}(\theta,\varepsilon)\times B_{H^+}(\theta,\epsilon)\times B_{H^-}(\theta,\epsilon)$. This, (3.29) and (6.12) lead to the conclusion of Claim 6.2.

Summarizing the above arguments we have proved the following theorem under the assumptions (6.7) and (6.13).

**Theorem 6.1.** *Let* $(H,X,\mathcal{L},A,B=P+Q)$ *and* $(\widehat{H},\widehat{X},\widehat{\mathcal{L}},\widehat{A},\widehat{B}=\widehat{P}+\widehat{Q})$ *be two tuples satisfying the conditions* (S), (F1) − (F3), (C1) − (C2) *and* (D) *in Section 2. Suppose that* $J:H\to\widehat{H}$ *is a linear injection satisfying (6.1)-(6.3). If* $\mu(\mathcal{L},\theta)=\mu(\widehat{\mathcal{L}},\theta)$ *then for the continuous maps* $h:B_{H^0}(\theta,\epsilon)\to X^\pm$ *and* $\hat{h}:B_{\widehat{H}^0}(\theta,\epsilon)\to\widehat{X}^\pm$*, and the origin-preserving homeomorphisms constructed in Theorem 2.1,*

$$\begin{aligned}&\Phi:B_{H^0}(\theta,\epsilon)\times(B_{H^+}(\theta,\epsilon)+B_{H^-}(\theta,\epsilon))\to W,\\ &\widehat{\Phi}:B_{\widehat{H}^0}(\theta,\epsilon)\times\big(B_{\widehat{H}^+}(\theta,\epsilon)+B_{\widehat{H}^-}(\theta,\epsilon)\big)\to\widehat{W},\end{aligned}$$

*it holds that*

$$\hat{h}(Jz)=J\circ h(z)\quad\textit{and}\quad\widehat{\Phi}(Jz,Ju^++Ju^-)=J\circ\Phi(z,u^++u^-)$$

*for all* $(z,u^+,u^-)\in B_{H^0}(\theta,\epsilon)\times B_{H^+}(\theta,\epsilon)\times B_{H^-}(\theta,\epsilon)$*. Consequently,*

$$\begin{aligned}&\widehat{\mathcal{L}}\circ\widehat{\Phi}(Jz,Ju^++Ju^-)=\mathcal{L}\circ\Phi(z,u^++u^-),\\ &\widehat{\mathcal{L}}(Jz+\hat{h}(Jz))=\mathcal{L}(z+h(z))\end{aligned}$$

*for all* $(z,u^+,u^-)\in B_{H^0}(\theta,\epsilon)\times B_{H^+}(\theta,\epsilon)\times B_{H^-}(\theta,\epsilon)$.

Here we understand $B_{H^0}(\theta,\epsilon)\times B_{H^+}(\theta,\epsilon)\times B_{H^-}(\theta,\epsilon)$ as $B_{H^0}(\theta,\epsilon)\times B_{H^+}(\theta,\epsilon)$ if $\dim H^-=0$, and $B_{H^0}(\theta,\epsilon)\times B_{H^+}(\theta,\epsilon)\times B_{H^-}(\theta,\epsilon)$ as $B_{H^-}(\theta,\epsilon)\times B_{H^+}(\theta,\epsilon)$ if $\dim H^0=0$.

Let us prove the remainder cases. Firstly, consider the case $\nu(\mathcal{L},\theta)=\nu(\widehat{\mathcal{L}},\theta)=0$. We only need to remove $z$ and $\hat{z}$ in the arguments below Claim 6.2 and then replace $\mathcal{F}$ and $\widehat{\mathcal{F}}$ by $\mathcal{L}$ and $\widehat{\mathcal{L}}$, respectively.

Finally, the case $0=\nu(\mathcal{L},\theta)<\nu(\widehat{\mathcal{L}},\theta)$ can also be obtained by combing the above three cases. Theorem 6.1 is proved.

By (6.3) and (6.9), for any $z\in B_{H^0}(\theta,r_0)$ it holds that

$$\widehat{\mathcal{L}}^\circ(Jz)=\widehat{\mathcal{L}}(Jz+\hat{h}(Jz))=\mathcal{L}(z+h(z))=\mathcal{L}^\circ(z).\tag{6.14}$$

**Corollary 6.2.** *Let* $(H,X,\mathcal{L},A,B=P+Q)$ *and* $(\widehat{H},\widehat{X},\widehat{\mathcal{L}},\widehat{A},\widehat{B}=\widehat{P}+\widehat{Q})$ *be two tuples satisfying the conditions* (S), (F1) $-$ (F3), (C1) $-$ (C2) *and* (D) *in Section 2. Suppose that* $J:H\to\widehat{H}$ *is a linear injection satisfying (6.1)-(6.3). If* $\nu(\mathcal{L},\theta)=\nu(\widehat{\mathcal{L}},\theta)>0$ *then*

$$C_q(\widehat{\mathcal{L}}^\circ,\theta;\mathbf{K})=C_q(\mathcal{L}^\circ,\theta;\mathbf{K})\quad\forall q\in\mathbb{N}\cup\{0\}.$$

**Theorem 6.3.** *Under the assumptions of Theorem 2.1, let* $(\widehat{H},\widehat{X})$ *be another pair of Hilbert-Banach spaces satisfying* **(S)***, and let* $J:H\to\widehat{H}$ *be a Hilbert space isomorphism which can induce a Banach space isomorphism* $J_X:X\to X$ *(this means that* $J(X)\subset X$ *and* $J|_X:X\to X$ *is a Banach space isomorphism). Set* $\widehat{V}=J(V)$ *(and hence* $\widehat{V}^{\widehat{X}}:=\widehat{V}\cap\widehat{X}=J(V^X)$*) and* $\widehat{\mathcal{L}}:\widehat{V}\to\mathbb{R}$ *by* $\widehat{\mathcal{L}}=\mathcal{L}\circ J^{-1}$*. Then* $(\widehat{H},\widehat{X},\widehat{V},\widehat{\mathcal{L}})$ *satisfies the assumptions of Theorem 2.1 too.*

*Proof.* Define $\widehat{A}:\widehat{V}^{\hat{X}}\to\widehat{X}$ by $\widehat{A}=J_X\circ A\circ J_X^{-1}$, and $\widehat{B}:\widehat{V}^{\hat{X}}\to\mathcal{L}_s(\widehat{H})$ by $\widehat{B}(\hat{x})=J\circ B(J_X^{-1}\hat{x})\circ J^{-1}$. Similarly, we also define $\widehat{P}(\hat{x})=J\circ P(J_X^{-1}\hat{x})\circ J^{-1}$ and $\widehat{Q}(\hat{x})=J\circ Q(J_X^{-1}\hat{x})\circ J^{-1}$. It is not hard to check that $(\widehat{H},\widehat{X},\widehat{V},\widehat{\mathcal{L}},\widehat{A},\widehat{B}=\widehat{P}+\widehat{Q})$ satisfies the assumptions of Theorem 2.1. □

**Theorem 6.4.** *Under the assumptions of Theorem 2.1, suppose that* $\check{H}\subset H$ *is a Hilbert subspace whose orthogonal complementary in* $H$ *is finite dimensional and is contained in* $X$*. Then* $(\mathcal{L}|_{\check{H}},\check{H},\check{X})$ *with* $\check{X}:=X\cap\check{H}$ *also satisfies the assumptions of Theorem 2.1 for the critical point* $\theta\in\check{H}$*.*

*Proof.* Let $P_{\check{H}}$ be the orthogonal projection onto $\check{H}$. Then $x-P_{\check{H}}x\in X\ \forall x\in X$ by the assumption $\check{H}^\perp\subset X$. It follows that $\check{A}(x):=P_{\check{H}}A(x)\in\check{X}$ for $x\in V^{\check{X}}:=V^X\cap\check{X}$. Since $\check{H}^\perp\subset X$ and $\dim\check{H}^\perp<\infty$, $P_{\check{H}}$ restricts to a bounded linear operator from $\check{X}$ to $\check{X}$. This implies that $\check{A}:V^{\check{X}}\to\check{X}$ has the same differentiability as $A$. It is easily checked that $D\mathcal{L}|_{\check{H}}(x)(u)=(\check{A}(x),u)_H\ \forall u\in\check{X}$, and that

$$(D\check{A}(x)(u),v)_H=(P_{\check{H}}DA(x)(u),v)_H=(P_{\check{H}}B(x)(u),v)_H=(\check{B}(x)u,v)_H$$

for any $x\in V^{\check X}$, $u,v\in\check X$, where $\check B(x):=P_{\check H}B(x)|_{\check H}\in\mathcal{L}_s(\check H)$. Obverse that

$$\begin{aligned}\|\check B(x_1)-\check B(x_2)\|_{\mathcal{L}_s(\check H)} &= \sup\{\|\check B(x_1)u-\check B(x_2)u\|_{\check H}: u\in\check H, \|u\|=1\}\\ &\le \|B(x_1)-B(x_2)\|_{\mathcal{L}_s(H)}\end{aligned}$$

for any $x_1,x_2\in V^{\check X}$. So some kind of continuality of $B$ implies the same continuous property of $\check B$. Suppose that $\check B(0)u=v$ for some $u\in\check H$ and $v\in\check X$. Then $P_{\check H}B(0)u=v$ and therefore $B(0)u=v+P_{\check H^\perp}B(0)u\in X$ because $P_{\check H^\perp}(\check H)=\check H^\perp\subset X$ by the assumptions. It follows that $u\in X$ and hence $u\in X\cap\check H=\check X$. That is, **(C2)** is satisfied. Since the eigenvectors of $\check B(0)$ are those of $B(0)$ too the condition **(D1)** holds naturally. For $x\in V\cap\check X$ take $\check P(x)=P_{\check H}\circ P(x)|_{\check H}$ and $\check Q(x)=P_{\check H}\circ Q(x)|_{\check H}$. It is also clear that $\check B(x)=\check P(x)+\check Q(x)$ satisfies the other conditions in **(D)**. □

# 7 An estimation for behavior of $\mathcal{L}$

In this section we shall estimate behavior of $\mathcal{L}$ near $\theta$. Such a result will be used in the proof of Theorem 5.1 of [34].

We shall replace the condition (**D4**) in Section 2 by the following stronger

**(D4**)** There exist positive constants $\eta_0'$ and $C_2'>C_1'$ such that

$$C_2'\|u\|^2\ge(P(x)u,u)\ge C_1'\|u\|^2\quad\forall u\in H,\ \forall x\in B_H(\theta,\eta_0')\cap X.$$

Note that $B(\theta)|_{H^\pm}:H^\pm\to H^\pm=H^-\oplus H^+$ is invertible. Set

$$\left.\begin{array}{l}B_\rho^{(*)}=\{h\in H^*\,|\,\|h\|\le\rho\},\ *=+,0,-,\\ B^\pm_{(r,s)}=B_r^{(-)}\oplus B_s^{(+)}.\end{array}\right\}$$

For the neighborhood $U$ in Lemma 3.4 we fix a small $\rho\in(0,\eta_0')$ so that

$$B_\rho^{(0)}\oplus B_\rho^{(-)}\oplus B_\rho^{(+)}\subset U.$$

We may assume that $a_1$ is no more than $a_0$ in Lemma 3.4. Set

$$a_1':=\frac{(2C_2'+\|Q(\theta)\|+1)}{2}+\frac{1}{3a_1}.\tag{7.1}$$

Since $h(\theta)=\theta$ we can choose $\rho_0\in(0,\rho]$ so small that $\omega$ in Lemma 3.3 and $Q$ in (**D3**) satisfy

$$\|Q(z+h(z)+u)-Q(\theta)\|\le\frac{C_1'}{2},\tag{7.2}$$

$$\omega(z+h(z)+u)<\sqrt{\frac{a_1}{2}},\tag{7.3}$$

$$\omega(z+h(z)+u)\le\frac{k}{8a_1'}\tag{7.4}$$

for all $z \in B^{(0)}_{\rho_0}$ and $u \in B^{\pm}_{(\rho_0,\rho_0)} \cap X$. As before we write $B_{H^\pm}(\theta,\delta)\cap X$ as $B_{H^\pm}(\theta,\delta)^X$ when it is considered as open subset of $X^\pm$, and $F^X$ as the restriction of the functional $F$ in (3.12) to $\bar{B}_{H^0}(\theta,\delta)\times B_{H^\pm}(\theta,\delta)^X$.

**Proposition 7.1.** *Under the assumptions of Theorem 2.1 with (***D4***) replaced by (***D4****), suppose that the map $A : V^X \to X$ in the condition (***F2***) is Fréchet differentiable. (This implies that the functional $B_{H^\pm}(\theta,\delta)^X \ni u \to F^X(z,u)$ is twice Fréchet differentiable for each fixed $z$). Let $s, r \in (0,\rho_0]$ satisfy*

$$B^{\pm}_{(r,s)} \subset B^{\pm}_{(\rho_0,\rho_0)} \quad \textit{for} \quad r = s\sqrt{\frac{8a_1'}{a_1}}. \tag{7.5}$$

*Then for positive constants*

$$\varepsilon = a_1' s^2 \quad \textit{and} \quad \hbar = \frac{a_1}{8}s^2 \tag{7.6}$$

*the following conclusions hold.*

(i) $(\nabla_2 F(z,u), P^+u) \ge \hbar \quad \forall (z,u) \in B^{(0)}_{\rho_0}\times B^{\pm}_{(r,s)}$ *with* $\|P^+u\| = s$*;*

(ii) $(\nabla_2 F(z,u), P^-u) \le -\hbar \quad \forall\,(z,u) \in B^{(0)}_{\rho_0}\times B^{\pm}_{(r,s)}$ *with* $F(z,u) = -\varepsilon$*;*

(iii) $F(z,u) \le -\varepsilon \quad \forall (z,u) \in B^{(0)}_{\rho_0}\times B^{\pm}_{(r,s)}$ *with* $\|P^-u\| = r$*.*

*In particular, taking $z = 0$ we get*

- $(\nabla\mathcal{L}(u), P^+u) \ge \hbar \quad \forall u \in B^+_s$ *with* $\|P^+u\| = s$,
- $(\nabla\mathcal{L}(u), P^-u) \le -\hbar \quad \forall\, u \in B^{\pm}_{(r,s)}$ *with* $\mathcal{L}(u) = -\varepsilon$;
- $\mathcal{L}(u) \le -\varepsilon \quad \forall u \in B^-_r$ *with* $\|P^-u\| = r$.

*Proof.* For $u \in B^{\pm}_{(\rho,\rho)} \cap X^\pm \setminus \{0\}$, since $H^-\oplus H^0 \subset X$, $P^+u = u - P^-u \in X^\pm$. Hence

$$\begin{aligned}
& (\nabla_2 F(z,u), P^+u) \\
= \;& d_uF(z,u)(P^+u) \\
= \;& d\mathcal{L}(z+h(z)+u)(P^+u) \\
= \;& d(\mathcal{L}|_X)(z+h(z)+u)(P^+u) \\
= \;& d(\mathcal{L}|_X)(u)(z+h(z)+u)(P^+u) - d(\mathcal{L}|_X)(z+h(z))(P^+u) \\
= \;& d^2(\mathcal{L}|_X)(z+h(z)+tu)(u, P^+u) \\
= \;& (B(z+h(z)+tu)u, P^+u) \\
= \;& (B(z+h(z)+tu)P^+u, P^+u) + (B(z+h(z)+tu)P^-u, P^+u)
\end{aligned}$$

for some $t \in (0,1)$. Here the fourth equality is because

$$\begin{aligned} d(\mathcal{L}|_X)(z+h(z))(P^+u) &= (A(z+h(z)), P^+u)_H \\ &= ((I-P^0)A(z+h(z)), P^+u)_H = 0, \end{aligned}$$

and the fifth equality comes from the mean value theorem. It follows from (i)-(ii) in Lemma 3.4 that

$$(\nabla_2 F(z,u), P^+u) \geq a_1\|P^+u\|^2 - \omega(z+h(z)+tu)\|P^-u\|\cdot\|P^+u\|.$$

Since $2pq \leq p^2+q^2$ for any $p,q\in\mathbb{R}$, we deduce that

$$\begin{aligned} & \omega(z+h(z)+tu)\|P^-u\|\cdot\|P^+u\| \\ = \;& 2\omega(z+h(z)+tu)\|P^-u\|\frac{1}{2\sqrt{\eta}}\sqrt{\eta}\|P^+u\| \\ \leq \;& \frac{1}{4\eta}(\omega(z+h(z)+tu)\|P^-u\|)^2 + \eta\|P^+u\|^2 \end{aligned}$$

for any $\eta > 0$. Taking $\eta = 3a_1/4$, we arrive at

$$(\nabla_2 F(z,u), P^+u) \geq \frac{a_1}{4}\|P^+u\|^2 - \frac{1}{3a_1}(\omega(z+h(z)+tu)\|P^-u\|)^2 \tag{7.7}$$

for all $u \in B^{\pm}_{(\rho,\rho)} \cap X^{\pm}\setminus\{0\}$, where $t = t(u)\in(0,1)$.

Similarly, for any $u \in B^{\pm}_{(\rho,\rho)} \cap X^{\pm}\setminus\{0\}$ and some $t' = t'(u)\in(0,1)$, we have

$$\begin{aligned} & (\nabla_2 F(z,u), P^-u) \\ = \;& d_uF(z,u)(P^-u) \\ = \;& d\mathcal{L}(z+h(z)+u)(P^-u) \\ = \;& d(\mathcal{L}|_X)(z+h(z)+u)(P^-u) \\ = \;& d(\mathcal{L}|_X)(z+h(z)+u)(P^-u) - d(\mathcal{L}|_X)(z+h(z))(P^-u) \\ = \;& d^2(\mathcal{L}|_X)(z+h(z)+t'u)(u, P^-u) \\ = \;& (B(z+h(z)+t'u)u, P^-u) \\ = \;& (B(z+h(z)+t'u)P^-u, P^-u) + (B(z+h(z)+t'u)P^+u, P^-u). \end{aligned}$$

Since for any $\eta > 0$,

$$\begin{aligned} & \omega(z+h(z)+t'u)\|P^+u\|\cdot\|P^-u\| \\ = \;& 2\omega(z+h(z)+t'u)\|P^+u\|\frac{1}{2\sqrt{\eta}}\sqrt{\eta}\|P^-u\| \\ \leq \;& \frac{1}{4\eta}(\omega(z+h(z)+t'u)\|P^+u\|)^2 + \eta\|P^-u\|^2, \end{aligned}$$

taking $\eta = 3a_1/4$, as above we derive from (ii)-(iii) of Lemma 3.4 that

$$\begin{aligned} & (\nabla_2 F(z,u), P^- u) \\ \le\ & -a_1\|P^- u\|^2 + \omega(z+h(z)+t'u)\|P^+ u\|\cdot\|P^- u\| \\ \le\ & -\frac{a_1}{4}\|P^- u\|^2 + \frac{1}{3a_1}(\omega(z+h(z)+t'u)\|P^+ u\|)^2. \end{aligned} \tag{7.8}$$

Since the functional $B_{H^\pm}(\theta,\delta)^X \ni u \to F^X(z,u)$ is twice Fréchet differentiable for each fixed $z$, by the Taylor formula, for $u \in B^{\pm}_{(\rho_0,\rho_0)} \cap X \setminus \{\theta\}$,

$$\begin{aligned} F(z,u) &= F(z,\theta) + \frac{1}{2}d_u^2 F^X(z,t''u)(u,u) \\ &= \frac{1}{2}d^2(\mathcal{L}|_X)(z+h(z)+t''u)(u,u) \\ &= \frac{1}{2}(B(z+h(z)+t''u)u,u) \\ &= \frac{1}{2}(B(z+h(z)+t''u)P^- u, P^- u) \\ &\quad +(B(z+h(z)+t''u)P^- u, P^+ u) \\ &\quad +\frac{1}{2}(B(z+h(z)+t''u)P^+ u, P^+ u) \end{aligned} \tag{7.9}$$

for some $t'' = t''(u) \in (0,1)$. As in the proof of (7.8) we have

$$\begin{aligned} & \frac{1}{2}(B(z+h(z)+t''u)P^- u, P^- u) + (B(z+h(z)+t''u)P^- u, P^+ u) \\ & \le -\frac{a_1}{2}\|P^- u\|^2 + \omega(z+h(z)+t''u)\|P^+ u\|\cdot\|P^- u\| \\ & \le -\frac{a_1}{4}\|P^- u\|^2 + \frac{1}{a_1}(\omega(z+h(z)+t''u)\|P^+ u\|)^2 \\ & \le -\frac{a_1}{4}\|P^- u\|^2 + \frac{1}{2}\|P^+ u\|^2 \end{aligned} \tag{7.10}$$

by (7.3). In addition, Since $C_1' < C_2'$, by the condition (**D4****) and (7.2)-(7.3),

$$\begin{aligned} & (B(z+h(z)+t''u)P^+ u, P^+ u) \\ =\ & (P(z+h(z)+t''u)P^+ u, P^+ u) + (Q(z+h(z)+t''u)P^+ u, P^+ u) \\ \le\ & C_2'\|P^+ u\|^2 + (C_2' + \|Q(\theta)\|)\|P^+ u\|^2. \end{aligned}$$

From this and (7.9)-(7.10) it follows that for any $(z,u) \in B^{(0)}_{\rho_0} \times (B^{\pm}_{(\rho_0,\rho_0)} \cap X)$,

$$F(z,u) \le -\frac{a_1}{4}\|P^- u\|^2 + \frac{(2C_2' + \|Q(\theta)\| + 1)}{2}\|P^+ u\|^2. \tag{7.11}$$

As in the proof of (7.7) we have

$$\begin{aligned} & \frac{1}{2}(B(z+h(z)+t''u)P^+ u, P^+ u) + (B(z+h(z)+t''u)P^- u, P^+ u) \\ & \ge \frac{a_1}{2}\|P^+ u\|^2 - \omega(z+h(z)+t''u)\|P^- u\|\cdot\|P^+ u\| \\ & \ge \frac{a_1-\eta}{2}\|P^+ u\|^2 - \frac{1}{2\eta}(\omega(z+h(z)+t''u)\|P^- u\|)^2 \end{aligned} \tag{7.12}$$

for any $0 < \eta < a_1$ because

$$\omega(z+h(z)+t''u)\|P^-u\|\cdot\|P^+u\| \le \frac{\eta}{2}\|P^+u\|^2 + \frac{1}{2\eta}(\omega(z+h(z)+t''u)\|P^-u\|)^2.$$

Note that the condition (**D4****) and (7.3) imply

$$\begin{aligned}
& (B(z+h(z)+t''u)P^-u, P^-u) \\
= \; & (P(z+h(z)+t''u)P^-u, P^-u) + (Q(z+h(z)+t''u)P^-u, P^-u) \\
\ge \; & C_1'\|P^-u\|^2 + (Q(z+h(z)+t''u)P^-u, P^-u) \\
\ge \; & C_1'\|P^-u\|^2 + (-\frac{C_1}{2} - \|Q(\theta)\|)\|P^-u\|^2 \\
= \; & \left(\frac{C_1'}{2} - \|Q(\theta)\|\right)\|P^-u\|^2.
\end{aligned}$$

From this, (7.9) and (7.12) we derive

$$F(z,u) \ge \frac{a_1-\eta}{2}\|P^+u\|^2 - \left[\frac{a_1}{4\eta} - \frac{C_1'}{4} + \frac{\|Q(\theta)\|}{2}\right]\|P^-u\|^2 \tag{7.13}$$

for all $(z,u) \in B^{(0)}_{\rho_0} \times (B^{\pm}_{(\rho_0,\rho_0)} \cap X)$.

Let us take $\eta$ such that

$$\frac{a_1}{4\eta} = \frac{C_1'}{4} + C_2' + \frac{1}{2}$$

Then $0 < \eta < a_1/8$, and by (7.1)

$$a_1' = \frac{(2C_2' + \|Q(\theta)\| + 1)}{2} + \frac{1}{3a_1} = \left[\frac{a_1}{4\eta} - \frac{C_1'}{4} + \frac{\|Q(\theta)\|}{2}\right] + \frac{1}{3a_1}.$$

It follows from (7.11) and (7.13) that

$$\frac{a_1}{4}\|P^+u\|^2 - a_1'\|P^-u\|^2 \le F(z,u) \le -\frac{a_1}{4}\|P^-u\|^2 + a_1'\|P^+u\|^2$$

for any $(z,u) \in B^{(0)}_{\rho_0} \times (B^{\pm}_{(\rho_0,\rho_0)} \cap X)$. This implies

$$\frac{a_1}{4}\|P^+u\|^2 - a_1'\|P^-u\|^2 \le F(z,u) \le -\frac{a_1}{4}\|P^-u\|^2 + a_1'\|P^+u\|^2 \tag{7.14}$$

for all $(z,u) \in B^{(0)}_{\rho_0} \times B^{\pm}_{(\rho_0,\rho_0)}$ because $B^{(0)}_{\rho_0} \times (B^{\pm}_{(\rho_0,\rho_0)} \cap X)$ is dense in $B^{(0)}_{\rho_0} \times B^{\pm}_{(\rho_0,\rho_0)}$.

Moreover, since $a_1' > \frac{1}{3a_1}$, by (7.7) and (7.8), for any $(z,u) \in B^{(0)}_{\rho_0} \times (B^{\pm}_{(\rho_0,\rho_0)} \cap X)$ with $u \ne 0$ there exist $t = t(u) \in (0,1)$ and $t' = t'(u) \in (0,1)$ such that

$$(\nabla_2 F(z,u), P^+u) \ge \frac{a_1}{4}\|P^+u\|^2 - a_1'(\omega(z+h(z)+tu))^2\|P^-u\|^2, \tag{7.15}$$

$$(\nabla_2 F(z,u), P^-u) \le -\frac{a_1}{4}\|P^-u\|^2 + a_1'(\omega(z+h(z)+t'u))^2\|P^+u\|^2. \tag{7.16}$$

Now we may prove that the positive constants $r$, $s$, $\varepsilon$ and $\hbar$ in (7.5)-(7.6) satisfy (i)-(iii). Firstly, for any $(z,u)\in B^{(0)}_{\rho_0}\times B^{\pm}_{(r,s)}$ with $\|P^-u\|=r$ it follows from (7.14) that

$$F(z,u)\le -\frac{a_1}{4}\|P^-u\|^2+a_1'\|P^+u\|^2\le -\frac{a_1}{4}r^2+a_1's^2=-a_1's^2=-\varepsilon.$$

Next, by (7.15) and (7.4) we have

$$(\nabla_2F(z,u),P^+u)\ge \frac{a_1}{4}\|P^+u\|^2-\frac{a_1^2}{64a_1'}\|P^-u\|^2$$

for any $(z,u)\in B^{(0)}_{\rho_0}\times(B^{\pm}_{(\rho_0,\rho_0)}\cap X)$. The density of $B^{(0)}_{\rho_0}\times(B^{\pm}_{(\rho_0,\rho_0)}\cap X)$ in $B^{(0)}_{\rho_0}\times B^{\pm}_{(\rho_0,\rho_0)}$ implies that this inequality also holds for any $(z,u)\in B^{(0)}_{\rho_0}\times B^{\pm}_{(\rho_0,\rho_0)}$. So

$$\begin{aligned}(\nabla_2F(z,u),P^+u) &\ge \frac{a_1}{4}\|P^+u\|^2-\frac{a_1^2}{64a_1'}\|P^-u\|^2\\ &\ge \frac{a_1}{4}s^2-\frac{a_1^2}{64a_1'}r^2=\frac{a_1}{8}s^2=\hbar\end{aligned}$$

for any $(z,u)\in B^{(0)}_{\rho_0}\times B^{\pm}_{(r,s)}$ with $\|P^+u\|=s$.

Finally, for any $(z,u)\in(B^{(0)}_{\rho_0}\times B^{\pm}_{(r,s)})\cap\{F(z,u)\le-\varepsilon\}$, by (7.14) we get

$$\frac{a_1}{4}\|P^+u\|^2-a_1'\|P^-u\|^2\le-\varepsilon. \tag{7.17}$$

This implies $a_1'\|P^-u\|^2\ge\varepsilon$, and thus $u\ne0$. If this $u$ also belongs to $X$, then it follows from this, (7.16) and (7.4) that

$$\begin{aligned}(\nabla_2F(z,u),P^-u) &\le -\frac{a_1}{4}\|P^-u\|^2+a_1'(\omega(z+h(z)+t'u))^2\|P^+u\|^2\\ &\le -\frac{a_1}{4}\|P^-u\|^2+\frac{a_1^2}{64a_1'}\|P^+u\|^2 \qquad \text{by } (7.4)\\ &\le -\frac{a_1}{4}\|P^-u\|^2+\frac{a_1^2}{64a_1'}\frac{4}{a_1}\Big[a_1'\|P^-u\|^2-\varepsilon\Big] \qquad \text{by } (7.17)\\ &\le -\frac{a_1}{4}\|P^-u\|^2+\frac{a_1}{16}\|P^-u\|^2-\frac{a_1\varepsilon}{16a_1'}\\ &= -\frac{3a_1}{16}\|P^-u\|^2-\frac{a_1\varepsilon}{16a_1'}\\ &\le -\frac{3a_1}{16}\frac{\varepsilon}{a_1'}-\frac{k\varepsilon}{16a_1'}=-\frac{a_1\varepsilon}{4a_1'}.\end{aligned}$$

Since $\big((B^{(0)}_{\rho_0}\times(B^{\pm}_{(r,s)})\cap X\big)\cap\{F(z,u)\le-\varepsilon\}$ is dense in $\big(B^{(0)}_{\rho_0}\times(B^{\pm}_{(r,s)})\big)\cap\{F(z,u)\le -\varepsilon\}$ we deduce that

$$(\nabla_2F(z,u),P^-u)\le-\frac{a_1\varepsilon}{4a_1'}<-\hbar$$

for all $(z,u)\in\big(B^{(0)}_{\rho_0}\times(B^{\pm}_{(r,s)})\big)\cap\{F(z,u)\le-\varepsilon\}$. $\square$

# 8 Concluding remarks

In this section we shall show that some conclusions of Theorem 2.1 can still be obtained if the strictly Fréchet differentiability at $\theta$ of the map $A : V^X \to X$ is replaced by a weaker condition similar to $(\mathrm{E}_\infty)$ or $(\mathrm{E}'_\infty)$ in Theorems 4.1 and 4.3 of [33]. That is, the condition (F2) can be replaced by the following weaker **(F2′)** or **(F2″)**.

**(F2′)** There exists a continuously directional differentiable (and thus $C^{1-0}$) map $A : V^X \to X$ such that $D\mathcal{L}(x)(u) = (A(x), u)_H$ for all $x \in V^X$ and $u \in X$ (which actually implies that $\mathcal{L}|_{V^X} \in C^1(V^X, \mathbb{R})$), and that

$$\begin{aligned}&\|(I-P^0)A(z_1+x_1) - B(\theta)x_1 - (I-P^0)A(z_2+x_2) + B(\theta)x_2\|_{X^\pm}\\ &\le \frac{1}{\kappa C_1}\|z_1+x_1-z_2-x_2\|_X \end{aligned} \tag{8.1}$$

for some positive numbers $\kappa > 1$, $r_1 > 0$ and all $z_i \in B_{H^0}(\theta, r_1)$, $x_i \in B_X(\theta, r_1) \cap X^\pm$, $i = 1, 2$. Here $C_1$ is given by (3.2).

**(F2″)** The inequality (8.1) in **(F2′)** is replaced by

$$\begin{aligned}&\|(I-P^0)A(z+x_1) - B(\theta)x_1 - (I-P^0)A(z+x_2) + B(\theta)x_2\|_{X^\pm}\\ &\le \frac{1}{\kappa C_1}\|x_1-x_2\|_X \end{aligned} \tag{8.2}$$

for some positive numbers $\kappa > 1$, $r_1 > 0$ and all $z \in B_{H^0}(\theta, r_1)$, $x_i \in B_X(\theta, r_1) \cap X^\pm$, $i = 1, 2$. Here $C_1$ is given by (3.2).

Clearly, (8.1) and (8.2) are, respectively, implied in the following inequalities

$$\begin{aligned}&\|A(z_1+x_1) - B(\theta)x_1 - A(z_2+x_2) + B(\theta)x_2\|_X\\ &\le \frac{1}{\kappa C_1 C_2}\|z_1+x_1-z_2-x_2\|_X \end{aligned} \tag{8.3}$$

for all $z_i \in B_{H^0}(\theta, r_1)$, $x_i \in B_X(\theta, r_1) \cap X^\pm$, $i = 1, 2$, and

$$\begin{aligned}&\|A(z+x_1) - B(\theta)x_1 - A(z+x_2) + B(\theta)x_2\|_X\\ &\le \frac{1}{\kappa C_1 C_2}\|x_1-x_2\|_X \end{aligned} \tag{8.4}$$

for all $z \in B_{H^0}(\theta, r_1)$, $x_i \in B_X(\theta, r_1) \cap X^\pm$, $i = 1, 2$. Here $C_1$ and $C_2$ are given by (3.2).

We first consider the case **(F2″)** holding. Checking the proof of (3.4) we have

$$\begin{aligned}&\|S(z,x_1) - S(z,x_2)\|_{X^\pm}\\ &\le C_1 \cdot \|(I-P^0)A(z+x_1) - B(\theta)x_1 - (I-P^0)A(z+x_2) + B(\theta)x_2\|_{X^\pm}\\ &\le \frac{1}{\kappa}\|x_1-x_2\|_X\end{aligned}$$

for all $z \in B_{H^0}(\theta, r_1)$ and $x_i \in B_{X^\pm}(\theta, r_1)$, $i = 1, 2$. Since $A(x) \to \theta$ as $x \to \theta$ we can choose $r_0 \in (0, r_1)$ such that $\|S(z, 0)\| < r_1(1 - 1/\kappa)$ for any $z \in B_{H^0}(\theta, r_0)$. By Theorem 10.1.1 in [18, Chap.10] we have a unique map $h : B_{H^0}(\theta, r_0) \to \bar{B}_{X^\pm}(\theta, r_0)$ with $h(\theta) = \theta$, which is also continuous, such that $S(z, h(z)) = h(z)$ or equivalently $(I - P^0)A(z + h(z)) = \theta \; \forall z \in B_{H^0}(\theta, r_0)$ as in (3.5).

Next we consider the case **(F2′)** holding. By the proof of (3.4) we easily see

$$\begin{aligned} &\|S(z_1, x_1) - S(z_2, x_2)\|_{X^\pm} \\ &\le C_1 \cdot \|(I - P^0)A(z_1 + x_1) - B(\theta)x_1 - (I - P^0)A(z_2 + x_2) + B(\theta)x_2\|_{X^\pm} \\ &\le \frac{1}{\kappa}\|z_1 + x_1 - z_2 - x_2\|_X \end{aligned} \tag{8.5}$$

and thus $\|S(z, x_1) - S(z, x_2)\|_{X^\pm} \le \frac{1}{\kappa}\|x_1 - x_2\|_X$ if $z_1 = z_2 = z$. Since $A(x) \to \theta$ as $x \to \theta$ we can choose $r_0 \in (0, r_1)$ such that

$$\begin{aligned} \|S(z, x)\|_{X^\pm} &= \|S(z, x) - S(z, \theta)\|_{X^\pm} + \|S(z, \theta)\| \\ &\le \frac{1}{\kappa}\|x\|_X + \frac{\kappa - 1}{\kappa} r_0 \end{aligned}$$

for any $z \in \bar{B}_{H^0}(\theta, r_0)$. Hence for each $z \in \bar{B}_{H^0}(\theta, r_0)$ we may apply the Banach fixed point theorem to the map

$$\bar{B}_{X^\pm}(\theta, r_0) \ni x \mapsto S(z, x) \in \bar{B}_{X^\pm}(\theta, r_0)$$

to get a unique map $h : \bar{B}_{H^0}(\theta, r_0) \to \bar{B}_{X^\pm}(\theta, r_0)$ such that $S(z, h(z)) = h(z)$. From the latter and (8.5) it easily follows that

$$\|h(z_1) - h(z_2)\|_{X^\pm} \le \frac{1}{\kappa - 1}\|z_1 - z_2\|_X \tag{8.6}$$

for any $z_i \in \bar{B}_{X^\pm}(\theta, r_0)$, $i = 1, 2$. That is, $h$ is Lipschitz continuous. Using this we may prove as in Step 2 of Lemma 3.1 that $\mathcal{L}^\circ$ has a linear bounded Gâteaux derivative at each $z_0 \in \bar{B}_{H^0}(\theta, r_0)$ and

$$D\mathcal{L}^\circ(z_0)z = (A(z_0 + h(z_0)), z)_H = (P^0 A(z_0 + h(z_0)), z)_H \; \forall z \in H^0.$$

Moreover, checking the proof of (3.10) we have still (3.10), i.e.,

$$\begin{aligned} |D\mathcal{L}^\circ(z_0)z - D\mathcal{L}^\circ(z_0')z| &\le \|A(z_0 + h(z_0)) - B(\theta)(z_0 + h(z_0)) \\ &\quad - A(z_0' + h(z_0')) + B(\theta)(z_0' + h(z_0'))\|_X \cdot \|z\|_X \end{aligned}$$

for all $z_0 \in \bar{B}_{H^0}(\theta, r_0)$ and $z \in H^0$. Note that $A$ is continuously directional differentiable and hence $C^{1-0}$. It follows from (8.6) that the map $\bar{B}_{H^0}(\theta, r_0) \ni z_0 \mapsto D\mathcal{L}^\circ(z_0) \in L(H^0, \mathbb{R})$ is $C^{1-0}$. As before we derive from [7, Th.2.1.13] that $\mathcal{L}^\circ$ is Fréchet differentiable at $z_0$ and its Fréchet differential $d\mathcal{L}^\circ(z_0) = D\mathcal{L}^\circ(z_0)$ is $C^{1-0}$ in $z_0 \in B_{H^0}(\theta, r_0)$.

Summarizing the above arguments we obtain

**Theorem 8.1.** *Under the above assumptions* (S), (F1),(**F2**$''$), (F3) *and* (C1)-(C2), (D), *if* $\nu > 0$ *there exist a positive* $\epsilon \in \mathbb{R}$, *a (unique) continuous map* $h : B_{H^0}(\theta, \epsilon) = B_H(\theta, \epsilon) \cap H^0 \to X^{\pm}$ *satisfying* $h(\theta) = \theta$ *and (2.3), an open neighborhood* $W$ *of* $\theta$ *in* $H$ *and an origin-preserving homeomorphism*

$$\Phi : B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon)) \to W$$

*of form* $\Phi(z, u^+ + u^-) = z + h(z) + \phi_z(u^+ + u^-)$ *with* $\phi_z(u^+ + u^-) \in H^{\pm}$ *such that (2.5) and (2.6) are satisfied. Moreover, the homeomorphism* $\Phi$ *has also the properties (a) and (b) in Theorem 2.1. Furthermore, if (*$\mathbf{F2}''$*) is replaced by the slightly strong (*$\mathbf{F2}'$*) then the map* $h$ *is Lipschitz continuous and the function* $B_{H^0}(\theta, \epsilon) \ni z \mapsto \mathcal{L}^\circ(z) := \mathcal{L}(z + h(z))$ *is* $C^{2-0}$ *and*

$$d\mathcal{L}^\circ(z_0)(z) = (A(z_0 + h(z_0)), z)_H \quad \forall z_0 \in B_{H^0}(\theta, \epsilon),\ z \in H^0.$$

*Consequently,* $\theta$ *is an isolated critical point of* $\mathcal{L}^\circ$ *provided that* $\theta$ *is an isolated critical point of* $\mathcal{L}|_{V^X}$.

Carefully checking the arguments in Section 2 and the proofs in Section 4 it is not hard to derive:

**Corollary 8.2.** *If the above assumptions* (S), (F1),(**F2**$''$), (F3) *and* (C1)-(C2), (D) *are satisfied then Corollary 2.5 also holds. Moreover, Corollaries 2.6, 2.7, 2.8 and 2.9 are true under the assumptions* (S), (F1),(**F2**$'$), (F3) *and* (C1)-(C2), (D).

By Claim 6.1, $\widehat{C}_1 := \|(\widehat{B}(\theta)|_{\widehat{X}^\pm})^{-1}\|_{L(\widehat{X}^\pm)} \ge C_1 := \|(B(\theta)|_{X^\pm})^{-1}\|_{L(X^\pm)}$ if $\|Jx\|_{\widehat{X}} = \|x\|_X\ \forall x \in X$. In order to assure that Theorem 6.1 also holds when Theorem 2.1 with (F2) is replaced Theorem 8.1 with ($\mathbf{F2}''$) we should require not only that $J|_X : X \to \widehat{X}$ is a Banach isometry but also that $C_1$ in (8.2) for $(A, B)$ is replaced by $\widehat{C}_1$. For Theorem 6.3 being true after Theorem 2.1 is replaced by Theorem 8.1 it is suffice to assume that $J|_X : X \to \widehat{X}$ is a Banach isometry. Theorem 6.4 also holds if we replace "Theorem 2.1" by "Theorem 8.1" there.

Finally, we have also a corresponding result with Proposition 7.1 provided that the sentence "Under the assumptions of Theorem 2.1 with (**D4**) replaced by (**D4****), we furthermore suppose that the map $A : V \cap X \to X$ in the condition (**F2**) is Fréchet differentiable." in Proposition 7.1 is replaced by "Under the assumptions of Theorem 8.1 with (**D4**) replaced by (**D4****), we furthermore suppose that the map $A : V \cap X \to X$ in the condition ($\mathbf{F2}''$) is Fréchet differentiable."

# A Parameterized version of Morse-Palais lemma due to Duc-Hung-Khai

Almost repeating the proof of Theorem 1.1 in [19] one easily gets the following parameterized version of it. Actually we give more conclusions, which are key for proofs of some results in this paper.

**Theorem A.1.** *Let $(H, \|\cdot\|)$ be a normed vector space and let $\Lambda$ be a compact topological space. Let $J : \Lambda \times B_H(\theta, 2\delta) \to \mathbb{R}$ be continuous, and for every $\lambda \in \Lambda$ the function $J(\lambda, \cdot) : B_H(\theta, 2\delta) \to \mathbb{R}$ is continuously directional differentiable. Assume that there exist a closed vector subspace $H^+$ and a finite-dimensional vector subspace $H^-$ of $H$ such that $H^+ \oplus H^-$ is a direct sum decomposition of $H$ and*

- **(i)** $J(\lambda, \theta) = 0$ *and* $D_2 J(\lambda, \theta) = 0$,
- **(ii)** $[D_2 J(\lambda, x + y_2) - D_2 J(\lambda, x + y_1)](y_2 - y_1) < 0$ *for any* $(\lambda, x) \in \Lambda \times \bar{B}_{H^+}(\theta, \delta)$, $y_1, y_2 \in \bar{B}_{H^-}(\theta, \delta)$ *and* $y_1 \neq y_2$,
- **(iii)** $D_2 J(\lambda, x + y)(x - y) > 0$ *for any* $(\lambda, x, y) \in \Lambda \times \bar{B}_{H^+}(\theta, \delta) \times \bar{B}_{H^-}(\theta, \delta)$ *and* $(x, y) \neq (\theta, \theta)$,
- **(iv)** $D_2 J(\lambda, x)x > p(\|x\|)$ *for any* $(\lambda, x) \in \Lambda \times \bar{B}_{H^+}(\theta, \delta) \setminus \{\theta\}$, *where* $p : (0, \delta] \to (0, \infty)$ *is a non-decreasing function.*

*Then there exist a positive $\epsilon \in \mathbb{R}$, an open neighborhood $U$ of $\Lambda \times \{\theta\}$ in $\Lambda \times H$ and a homeomorphism*

$$\phi : \Lambda \times \left(B_{H^+}(\theta, \sqrt{p(\epsilon)/2}) + B_{H^-}(\theta, \sqrt{p(\epsilon)/2})\right) \to U$$

*such that*

$$J(\lambda, \phi(\lambda, x + y)) = \|x\|^2 - \|y\|^2 \quad \textit{and} \quad \phi(\lambda, x + y) = (\lambda, \phi_\lambda(x + y)) \in \Lambda \times H$$

*for all $(\lambda, x, y) \in \Lambda \times B_{H^+}(\theta, \sqrt{p(\epsilon)/2}) \times B_{H^-}(\theta, \sqrt{p(\epsilon)/2})$. Moreover, for each $\lambda \in \Lambda$, $\phi_\lambda(0) = 0$, $\phi_\lambda(x + y) \in H^-$ if and only if $x = 0$, and $\phi$ is a homoeomorphism from $\Lambda \times B_{H^-}(\theta, \sqrt{p(\epsilon)/2})$ onto $U \cap (\Lambda \times H^-)$ according to the topologies on both induced by any norms on $H^-$.*

The claim in "Moreover" part was not stated in [19], and can be seen from the proof therein. It precisely means: for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $H^-$, if $\Lambda \times B_{H^-}(\theta, \sqrt{p(\epsilon)/2})$ (resp. $U \cap (\Lambda \times H^-)$) is equipped with the topology induced by $\Lambda \times (H^-, \|\cdot\|_1)$ (resp. $\Lambda \times (H^-, \|\cdot\|_2)$) then $\phi$ is also a homoeomorphism from $\Lambda \times B_{H^-}(\theta, \sqrt{p(\epsilon)/2})$ onto $U \cap (\Lambda \times H^-)$. This leads to the proof of Theorem 2.1(b), which is a key for the proofs of Corollary 2.5 and Theorem 2.10. So it is helpful for readers to outline the proof of Theorem A.1.

*Sketches of proof of Theorem A.1.* $1^\circ)$ **Case** $H^- = \{\theta\}$. This is actually contained in the proof of [19]. Since $H^+ = H$ the condition (ii) is trivial and (iii) is implied in (iv). Define

$$\psi(\lambda, x) = \begin{cases} \frac{\sqrt{J(\lambda,x)}}{\|x\|}x & \text{if } x \in \bar{B}_H(\theta,\delta) \setminus \{\theta\}, \\ \theta & \text{if } x = \theta. \end{cases}$$

Then it is continuous and $J(\lambda, x) = \|\psi(\lambda, x)\|^2$. It easily follows from the condition (iv) that for each $\lambda \in \Lambda$ the map $\psi(\lambda, \cdot)$ is one-to-one on $\bar{B}_H(\theta, \delta)$. Moreover, for any $x \in \partial B_H(\theta, \delta)$, as in [19, (2.9)] we have $s_x \in (1/2, 1)$ such that

$$\begin{aligned} &J(\lambda, x) > J(\lambda, x) - J(\lambda, x/2) = D_2 J(\lambda, s_x x)(x/2) \\ &= \frac{1}{2s_x} D_2 J(\lambda, s_x x)(s_x x) > \frac{1}{2} p(\|s_x x\|) \ge \frac{1}{2} p(\|x/2\|) = \frac{1}{2} p(\delta/2) \end{aligned}$$

by the condition (iv). Hence $\|\psi(\lambda, x)\| > \sqrt{p(\delta/2)/2}$. For any $0 < \|y\| < \sqrt{p(\delta/2)/2}$, without loss of generality we assume $\delta > \sqrt{p(\delta/2)/2}$. Then we have a unique positive number $r > 1$ such that $x := ry \in \partial B_H(\theta, \delta)$. Since the function

$$[0, 1] \to \mathbb{R},\ t \mapsto \sqrt{J(\lambda, tx)}$$

is continuous there exists a $t_0 \in (0, 1)$ such that $\|y\| = \sqrt{J(\lambda, t_0 x)}$ and hence

$$\psi(\lambda, t_0 x) = \sqrt{J(\lambda, t_0 x)} \frac{t_0 x}{\|t_0 x\|} = \|y\| \frac{y}{\|y\|} = y.$$

This shows that $B_H\big(\theta, \sqrt{p(\delta/2)/2}\big) \subset \psi\big(\{\lambda\} \times B_H(\theta, \delta)\big)$. Let

$$U = \Big\{(\lambda, z) \in \Lambda \times B_H(\theta, \delta)\ \Big|\ \psi(\lambda, z) \in B_H\big(\theta, \sqrt{p(\delta/2)/2}\big)\Big\}.$$

It is an open neighborhood of $\Lambda \times \{\theta\}$ in $\Lambda \times H$. Define

$$\phi : \Lambda \times B_H(\theta, \sqrt{p(\delta/2)/2}) \to U,\ (\lambda, x) \mapsto (\lambda, y),$$

where $y \in B_H(\theta, \delta)$ is the unique point such that $\psi(\lambda, y) = x$. As in the proof of Lemma 2.7 of [19] it is easily showed that $\phi$ is continuous and satisfies

$$J(\lambda, \phi(\lambda, x)) = \|x\|^2 \quad \forall (\lambda, x) \in \Lambda \times B_H(\theta, \sqrt{p(\delta/2)/2}).$$

$2^\circ)$ **Case** $0 < \dim H^- < \infty$. Since the parameter $\lambda$ appears many notations in [19] have corresponding changes.

**Step 1**([19, Lemma 2.1]). There exists a positive real number $\epsilon_1 < \delta$ having the following property: For each $(\lambda, x) \in \Lambda \times B_{H^+}(\theta, \epsilon_1)$ there exists a unique $\varphi_\lambda(x) \in B_{H^-}(\theta, \delta)$ such that

$$J(\lambda, x + \varphi_\lambda(x)) = \max\{J(\lambda, x + y) \,|\, y \in B_{H^-}(\theta, \delta)\}.$$

See the proof of Claim A.3 in the proof of Theorem A.2. (*Note*: The compactness of $\Lambda$ is necessary in proving this claim.)

Remarks that $J(\lambda, x+\varphi_\lambda(x)) > 0$ for any $x \in B_{H^+}(\theta, \delta) \setminus \{\theta\}$ by Theorem A.1(iv) and the mean value theorem. Moreover, the uniqueness of $\varphi_\lambda(x)$ implies

$$J(\lambda, x+\varphi_\lambda(x)) > J(\lambda, x+y)$$

for all $x \in B_{H^+}(\theta, \epsilon_1)$ and $y \in B_{H^-}(\theta, \delta) \setminus \{\varphi_\lambda(x)\}$.

By replacing $\delta$ by $\delta/2$ in the arguments above we can assume $\varphi_\lambda(x) \in B_{H^-}(\theta, \delta/2)$ for any $x \in B_{H^+}(\theta, \epsilon_1)$ below.

**Step 2**([19, Lemma 2.2]). The map $\Lambda \times B_{H^+}(\theta, \epsilon_1) : (\lambda, x) \mapsto \varphi_\lambda(x)$ is continuous.[11]

In fact, suppose that the sequence $\{(\lambda_n, x_n)\} \subset \Lambda(\mu) \times B_{H^+}(0, \epsilon_1)$ converges to $(\lambda_0, x_0) \in \Lambda(\mu) \times B_{H^+}(0, \epsilon_1)$. Since $\bar{B}_{H^-}(0, \delta/2)$ is compact, we can assume that $\{\varphi_{\lambda_n}(x_n)\}$ converges to $y_0 \in \bar{B}_{H^-}(0, \delta/2)$. Then

$$J(\lambda_n, x_n + \varphi_{\lambda_n}(x_n)) \geq J(\lambda_n, x_n + y) \quad \forall y \in B_{H^-}(0, \delta) \text{ and } n \in \mathbb{N}.$$

This implies that $J(\lambda_0, x_0+y_0) \geq J(\lambda_0, x_0+y)$ for any $y \in B_{H^-}(0, \delta)$. By the uniqueness of $\varphi_{\lambda_0}(x_0)$ we get $y_0 = \varphi_{\lambda_0}(x_0)$.

**Step 3**([19, Lemma 2.3]). Put $j(\lambda, x) = J(\lambda, x+\varphi_\lambda(x))$ for any $(\lambda, x) \in \Lambda \times B_{H^+}(\theta, \epsilon_1)$. Then $j$ is continuous and for each $\lambda \in \Lambda$ the map $x \mapsto j(\lambda, x)$ is continuously directional differentiable.

**Step 4**([19, Lemma 2.4]). Define

$$\begin{aligned}
&\psi_1(\lambda, x+y) = \begin{cases} \frac{\sqrt{J(\lambda, x+\varphi_\lambda(x))}}{\|x\|} x & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta, \end{cases} \\
&\psi_2(\lambda, x+y) = \begin{cases} \frac{\sqrt{J(\lambda, x+\varphi_\lambda(x)) - J(\lambda, x+y)}}{\|y-\varphi_\lambda(x)\|} (y - \varphi_\lambda(x)) & \text{if } y \neq \varphi_\lambda(x), \\ \theta & \text{if } y = \varphi_\lambda(x), \end{cases} \\
&\psi(\lambda, x+y) = \psi_1(\lambda, x+y) + \psi_2(\lambda, x+y) \\
&\qquad\qquad \forall (x, y) \in B_{H^+}(\theta, \epsilon_1) \times B_{H^-}(\theta, \delta).
\end{aligned}$$

Then $\psi_1$, $\psi_2$ and $\psi$ are continuous on $\Lambda \times (B_{H^+}(\theta, \epsilon_1) + B_{H^-}(\theta, \delta))$ and

$$J(\lambda, x+y) = \|\psi_1(\lambda, x+y)\|^2 - \|\psi_2(\lambda, x+y)\|^2 \tag{A.1}$$

for any $(\lambda, x, y) \in \Lambda \times B_{H^+}(\theta, \epsilon_1) \times B_{H^-}(\theta, \delta)$. Moreover, $\psi(\lambda, x+y) \in \mathrm{Im}(\psi) \cap H^-$ if and only if $x = \theta$.

**Step 5**([19, Lemma 2.5]). For each $\lambda \in \Lambda$ the map

$$\psi(\lambda, \cdot) : B_{H^+}(\theta, \epsilon_1) + B_{H^-}(\theta, \delta) \to H^{\pm}$$

[11] The local compactness of $\Lambda$ is sufficient for its proof.

is injective.

**Step 6**([19, Lemma 2.6]). There is a positive real number $\epsilon<\epsilon_1$ such that

$$B_{H^+}\big(\theta,\sqrt{p(\epsilon)/2}\big)+B_{H^-}\big(\theta,\sqrt{p(\epsilon)/2}\big)\subset\psi\big(\lambda,B_{H^+}(\theta,2\epsilon)+B_{H^-}(\theta,\delta)\big)$$

for any $\lambda\in\Lambda$.

We here give a detailed proof of it because the compactness of $\Lambda$ is very key in the following proof. They are helpful for understanding the proof of the noncompact case in Section 4 of [33].

For each $(\lambda,y)\in\Lambda\times\bar{B}_{H^-}(0,\delta)$ with $y\ne 0$, the mean value theorem yields $\bar{t}\in(0,1)$ such that

$$J(\lambda,y)=J(\lambda,y)-J(\lambda,0)=D_2J(\lambda,\bar{t}\cdot y)y=\frac{-1}{\bar{t}}D_2J(\lambda,\bar{t}\cdot y)(-\bar{t}\cdot y)<0$$

because of the condition (iii) in Theorem A.1. So the compactness of $\Lambda\times\partial B_{H^-}(0,\delta)$ implies that there exists a positive real number $C$ such that

$$J(\lambda,y)<-C\quad\forall(\lambda,y)\in\Lambda\times\partial B_{H^-}(0,\delta).\tag{A.2}$$

We shall prove that there exists a positive real number $\epsilon<\epsilon_1/4$ such that

$$J(\lambda,x+y)\le 0\quad\forall(\lambda,x,y)\in\Lambda\times\bar{B}_{H^+}(0,2\epsilon)\times\partial B_{H^-}(0,\delta).\tag{A.3}$$

Assume by contradiction that there exists a sequence

$$\{(\lambda_n,x_n,y_n)\}\subset\Lambda\times\bar{B}_{H^+}(0,\epsilon_1)\times\partial B_{H^-}(0,\delta)$$

such that $(\lambda_n,x_n,y_n)\to(\lambda_0,\theta,y_0)\in\Lambda\times\bar{B}_{H^+}(0,\epsilon_1)\times\partial B_{H^-}(0,\delta)$ and $J(\lambda_n,x_n+y_n)\ge 0\ \forall n$. Then the continuity of $J$ implies $J(\lambda_0,y_0)\ge 0$. This contradicts to (A.2). Hence (A.3) holds.

Since $\varphi_\lambda(0)=0\ \forall\lambda\in\Lambda$, by Step 2 we may shrink $\epsilon$ in (A.3) such that

$$\varphi_\lambda(\bar{B}_{H^+}(0,2\epsilon))\subset B_{H^-}(0,\delta/2)\quad\forall\lambda\in\Lambda.\tag{A.4}$$

Fixing $(\lambda,x)\in\Lambda\times\bar{B}_{H^+}(0,2\epsilon)\setminus\{0\}$ we can use the mean value theorem and the condition (iv) in Theorem A.1 to get $s_x\in(1/2,1)$ such that

$$\begin{aligned}J(\lambda,x+\varphi_\lambda(x))&\ge J(\lambda,x)>J(\lambda,x)-J(\lambda,x/2)\\&=D_2J(\lambda,s_xx)(x/2)\\&=\frac{1}{2s_x}D_2J(\lambda,s_xx)(s_xx)\\&>\frac{1}{2}p(\|s_xx\|)\ge\frac{1}{2}p(\|x/2\|)\end{aligned}\tag{A.5}$$

This and (A.3) imply that for any $(\lambda, x, y) \in \Lambda \times \partial B_{H^+}(0, 2\epsilon) \times \partial B_{H^-}(0, \delta)$,

$$\begin{aligned} J(\lambda, x + \varphi_\lambda(x)) - J(\lambda, x + y) &\geq J(\lambda, x + \varphi_\lambda(x)) \\ &> \frac{1}{2} p(\|x/2\|) = \frac{p(\epsilon)}{2}. \end{aligned} \tag{A.6}$$

Now for $x \in \partial B_{H^+}(0, 2\epsilon)$ and $0 \leq t \leq \sqrt{p(\epsilon)/2}$, by (A.5) we have

$$\sqrt{J(\lambda, x + \varphi_\lambda(x))} > \sqrt{p(\epsilon)/2} \geq t \geq 0.$$

Since the map $[0, 1] \to \mathbb{R}, s \mapsto J(\lambda, sx + \varphi_\lambda(sx))$, is continuous we may obtain a $\bar{s} \in [0, 1)$ such that $\sqrt{J(\lambda, \bar{s}x + \varphi_\lambda(\bar{s}x))} = t$. Clearly, $\bar{s} > 0$ if and only $t > 0$. If $t > 0$, by the definition of $\psi_1$ we get

$$\psi_1(\lambda, \bar{s}x + y) = \frac{t}{\|x\|} x = \frac{t}{\|\bar{s}x\|} \bar{s}x \quad \forall y \in B_{H^-}(0, \delta).$$

When $t = 0$, $\psi_1(\lambda, 0) = 0$. So for any $x \in \partial B_{H^+}(0, 2\epsilon)$ we have always

$$\left\{ \frac{t}{\|x\|} x \;\middle|\; 0 \leq t \leq \sqrt{p(\epsilon)/2} \right\} \subset \psi_1\big(\lambda, B_{H^+}(0, 2\epsilon)\big),$$

that is,

$$\bar{B}_{H^+}(0, \sqrt{p(\epsilon)/2}) \subset \psi_1\big(\lambda, B_{H^+}(0, 2\epsilon)\big) \quad \forall \lambda \in \Lambda. \tag{A.7}$$

For a given $(x^*, y^*) \in \bar{B}_{H^+}(0, \sqrt{p(\epsilon)/2}) \times \bar{B}_{H^-}(0, \sqrt{p(\epsilon)/2})$, we may assume $x^* \neq \theta$ and $y^* \neq \theta$, by (A.7) we have $x_\lambda \in B_{H^+}(0, 2\epsilon) \setminus \{\theta\}$ such that

$$\psi_1(\lambda, x_\lambda + y) = x^* \quad \forall y \in B_{H^-}(\theta, \delta). \tag{A.8}$$

Let us write $y^* = \bar{t}z/\|z\|$, where $z \in \partial B_{H^-}(0, \delta/2)$ and $0 < \bar{t} \leq \sqrt{p(\epsilon)/2}$. Since $\varphi_\lambda(x_\lambda) \in B_{H^-}(0, \delta/2)$ by (A.4), and $\varphi_\lambda(x_\lambda) \neq \theta$, we have always a real number $k$ with $|k| > 1$ such that

$$y := kz + \varphi_\lambda(x_\lambda) \in \partial B_{H^-}(0, \delta)$$

(because $|k \cdot z| = |y - \varphi_\lambda(x)| \geq |y| - |\varphi_\lambda(x)| > \delta/2$). By (A.6) the continuous map

$$[0, 1] \mapsto \mathbb{R},\; s \mapsto J(\lambda, x_\lambda + \varphi_\lambda(x_\lambda)) - J(\lambda, x + (1 - s)\varphi_\lambda(x_\lambda) + sy)$$

takes a value $J(\lambda, x_\lambda + \varphi_\lambda(x_\lambda)) - J(\lambda, y) > p(\epsilon)/2$ at $s = 1$, and zero at $s = 0$. So we have $\hat{s} \in (0, 1)$ such that

$$\sqrt{J(\lambda, x_\lambda + \varphi_\lambda(x_\lambda)) - J(\lambda, x + (1 - \hat{s})\varphi_\lambda(x_\lambda) + \hat{s}y)} = \bar{t}.$$

Set

$$\begin{aligned} y_\lambda := (1 - \hat{s})\varphi_\lambda(x_\lambda) + \hat{s}y &= (1 - \hat{s})\varphi_\lambda(x_\lambda) + \hat{s}k_z \cdot z + \hat{s}\varphi_\lambda(x_\lambda) \\ &= \varphi_\lambda(x_\lambda) + \hat{s}k_z \cdot z. \end{aligned}$$

Then

$$\|y_\lambda\| = \|(1-\hat{s})\varphi_\lambda(x_\lambda) + \hat{s}y\| \le (1-\hat{s})\|\varphi_\lambda(x_\lambda)\| + \hat{s}\delta < (1-\hat{s})\delta/2 + \hat{s}\delta < \delta,$$

and the definition of $\psi_2$ shows that

$$\psi_2(\lambda, x_\lambda + y_\lambda) = \frac{\bar{t}}{\|y_\lambda - \varphi_\lambda(x_\lambda)\|}(y_\lambda - \varphi_\lambda(x_\lambda)) = \frac{\bar{t}}{\|z\|}z = y^*.$$

This and (A.8) show that $\psi(\lambda, x_\lambda + y_\lambda) = (x^*, y^*)$. The desired result is proved. □
**Step 7**([19, Lemma 2.7]). Put

$$U = [\Lambda \times (B_{H^+}(\theta, 2\epsilon) + B_{H^-}(\theta, \delta))] \cap \psi^{-1}\left(B_{H^+}(\theta, \sqrt{p(\epsilon)/2}) + B_{H^-}(\theta, \sqrt{p(\epsilon)/2})\right)$$

and

$$\begin{aligned} \phi : \Lambda \times \left(B_{H^+}(\theta, \sqrt{p(\epsilon)/2}) + B_{H^-}(\theta, \sqrt{p(\epsilon)/2})\right) &\to U, \\ (\lambda, x + y) \mapsto (\lambda, \phi_\lambda(x+y)) &:= (\lambda, x' + y'), \end{aligned} \tag{A.9}$$

where $(x', y') \in B_{H^+}(\theta, 2\epsilon) \times B_{H^-}(\theta, \delta)$ is a unique point satisfying $x + y = \psi(\lambda, x' + y')$. Then $\phi$ is continuous and

$$J(\phi(\lambda, x + y)) = \|x\|^2 - \|y\|^2$$

for any $(\lambda, x, y) \in \Lambda \times B_{H^+}(\theta, \sqrt{p(\epsilon)/2}) \times B_{H^-}(\theta, \sqrt{p(\epsilon)/2})$. Moreover, $\phi(\lambda, x+y) \in \mathrm{Im}(\psi) \cap (\Lambda \times H^-)$ if and only if $x = \theta$.
**Step 8.** We shall prove the claims in "Moreover" part of Theorem A.1. It suffices to check Steps 4, 7. By Step 1, for each $(\lambda, x) \in \Lambda \times B_{H^+}(\theta, \epsilon_1)$, $\varphi_\lambda(x) \in B_{H^-}(\theta, \delta)$ is a unique maximum point of the function $B_{H^-}(\theta, \delta) \to \mathbb{R},\ y \mapsto J(\lambda, x+y)$. For any $y \in B_{H^-}(\theta, \delta)$ with $y \ne \theta$, it follows from the condition (ii) and the mean value theorem that

$$J(\lambda, y) = J(\lambda, y) - J(\lambda, \theta) = D_2J(\lambda, ty)(y) = \frac{1}{t}D_2J(\lambda, ty)(ty) < 0$$

for some $t \in (0, 1)$. Hence $\varphi_\lambda(\theta) = \theta$. For any $x \in B_{H^+}(\theta, \epsilon_1)$ with $x \ne \theta$, by the condition (iv) and the similar reason we get a $t \in (0, 1)$ such that

$$J(\lambda, x + \varphi(x)) \ge J(\lambda, x) - J(\lambda, \theta) = D_2J(\lambda, tx)(x) > p(\|tx\|)/t > 0.$$

This implies that $\psi_1(\lambda, x + y) \ne \theta$ if $x \ne \theta$. When $\psi(\lambda, x + y) \in H^-$, $\psi_1(\lambda, x + y) = \theta$ and thus $x = \theta$. Conversely, if $x = \theta$ then $\psi_1(y) = \theta$ and

$$\psi(\lambda, y) = \theta + \psi_2(\lambda, y) = \begin{cases} \frac{\sqrt{-J(\lambda, y)}}{\|y\|}y & \text{if } y \ne \theta, \\ \theta & \text{if } y = \theta. \end{cases}$$

Hence we get that $\psi(\lambda, x+y) \in H^-$ if and only if $x = \theta$. By the definition of $\phi$ in (A.9), it is easy to see that $\phi(\lambda, x+y)$ sits in $U \cap (\Lambda \times H^-)$ if and only if $x = \theta$.

As to the final claim, since $\dim H^- < \infty$ implies that any norm $\|\cdot\|^*$ on $H^-$ is equivalent to the original $\|\cdot\|$, $\Lambda \times (H^-, \|\cdot\|^*)$ and $\Lambda \times (H^-, \|\cdot\|)$ induce equivalent topologies on each one of the sets $\Lambda \times B_{H^-}(\theta, \sqrt{p(\epsilon)/2})$ and $U \cap (\Lambda \times H^-)$. The claim follows. $\square$

In order to give the corresponding version at critical submanifolds we need a more general result than Theorem A.1. For future conveniences we here present it because many arguments and notations can be saved. Let $\Lambda$ and $\mathcal{E}$ be two topological spaces. Imitating [29, §1 of Chap.III] one can naturally define a *topological normed vector bundle* over $\Lambda$ to be a triple $(\mathcal{E}, \Lambda, p)$, where $p : \mathcal{E} \to \Lambda$ is a continuous surjection (projection). In particular we have the notions of a *topological Banach* (resp. *Hilbert*) *vector bundle*. Corresponding to Definition 3.1 in Chapter 2 of [25], a *bundle morphism* from the normed vector bundles $p_1 : \mathcal{E}^{(1)} \to \Lambda_1$ to $p_2 : \mathcal{E}^{(2)} \to \Lambda_2$ is a pair of continuous maps $(\tilde{f}, f)$, where $\tilde{f} : \mathcal{E}^{(1)} \to \mathcal{E}^{(2)}$ and $f : \Lambda_1 \to \Lambda_2$ such that $p_2 \circ \tilde{f} = f \circ p_1$. As on the pages 43-44 of [29] we may define the notion of a *normed vector bundle morphism*. If $\Lambda_1 = \Lambda_2 = \Lambda$ and $f = id_\Lambda$ we get the notions of a $\Lambda$-*bundle morphism* and a $\Lambda$-*normed vector bundle morphism*. When $f$ and $\tilde{f}$ are homeomorphisms onto $\Lambda_2$ and $\mathcal{E}^{(2)}$ the corresponding bundle morphism and normed vector bundle morphism $(\tilde{f}, f)$ are called *bundle isomorphism* and *normed vector bundle isomorphism* from $\mathcal{E}^{(1)}$ onto $\mathcal{E}^{(2)}$. See [29] for more notions such as subbundles and so on. As in [11, Def.2.2, page 15] we can define a *Finsler structure* on the bundle $p : \mathcal{E} \to \Lambda$, and show the existence of such a structure on the vector bundle if $\Lambda$ is paracompact.

Let $G$ be a topological group. For a normed vector bundle $p : \mathcal{E} \to \Lambda$, let both $\mathcal{E}$ and $\Lambda$ be also $G$-spaces and let $p$ be a $G$-map (or $G$-equivariant map), we call it a $G$-*normed vector bundle* if for all $g \in G$ the action of $g : \mathcal{E}_\lambda \to \mathcal{E}_{g\lambda}$ is a vector space isomorphism.

**Theorem A.2.** *Let $\Lambda$ be a compact topological space, and let $p : \mathcal{E} \to \Lambda$ be a topological normed vector bundle with a Finsler structure $\|\cdot\| : \mathcal{E} \to [0, \infty)$. Suppose that $\mathcal{E}$ can be split into a direct sum of two topological normed vector subbundles, $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, where $p_- : \mathcal{E}^- \to \Lambda$ has finite rank. For $\delta > 0$ let $B_\delta(\mathcal{E}) = \{(\lambda, v) \in \mathcal{E}_\lambda \,|\, \|v\|_\lambda := \|(\lambda, v)\| < \delta\}$. Assume that $J : B_{2\delta}(\mathcal{E}) \to \mathbb{R}$ is continuous and that the restriction of it to each fiber*

$$J_\lambda : B_{2\delta}(\mathcal{E})_\lambda = \{v \in \mathcal{E}_\lambda \,|\, \|v\|_\lambda < 2\delta\},\ v \mapsto J(\lambda, v)$$

*is continuously directional differentiable. Furthermore assume:*

**(i)** $J_\lambda(\theta_\lambda) = 0$ *and* $DJ_\lambda(\theta_\lambda) = 0$,

**(ii)** $[DJ_\lambda(x+y_2) - DJ_\lambda(x+y_1)](y_2 - y_1) > 0$ *for any* $(\lambda, x) \in \bar{B}_\delta(\mathcal{E}^+)$, $y_1, y_2 \in \bar{B}_\delta(\mathcal{E}^-)$ *and* $y_1 \neq y_2$,

**(iii)** $DJ_\lambda(x+y)(x-y) > 0$ *for any* $(\lambda, x) \in \bar{B}_\delta(\mathcal{E}^+)$ *and* $(\lambda, y) \in \bar{B}_\delta(\mathcal{E}^+)$ *with* $x+y \neq \theta_\lambda$,

**(iv)** $DJ_\lambda(x)x > p(\|x\|_\lambda)$ *for any* $(\lambda, x) \in \bar{B}_\delta(\mathcal{E}^+)$ *with* $x \neq \theta_\lambda$, *where* $p : (0, \delta] \to (0, \infty)$ *is a non-decreasing function independent of* $\lambda \in \Lambda$.

*Then there exist a positive* $\epsilon \in \mathbb{R}$, *an open neighborhood* $U$ *of the zero section* $0_{\mathcal{E}}$ *of* $\mathcal{E}$ *and a homeomorphism*

$$\phi : B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^+) \oplus B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^-) \to U$$

*such that*

$$J(\phi(\lambda, x+y)) = \|x\|_\lambda^2 - \|y\|_\lambda^2 \quad \textit{and} \quad \phi(\lambda, x+y) = (\lambda, \phi_\lambda(x+y)) \in \mathcal{E}$$

*for all* $(\lambda, x+y) \in B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^+) \oplus B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^-)$. *Moreover, for each* $\lambda \in \Lambda$, $\phi_\lambda(\theta_\lambda) = \theta_\lambda$, $\phi_\lambda(x+y) \in \mathcal{E}_\lambda^-$ *if and only if* $x = \theta_\lambda$, *and* $\phi$ *is a homoeomorphism from* $B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^-)$ *onto* $U \cap \mathcal{E}^-$ *according to any topology on both induced by any Finsler structure on* $\mathcal{E}^-$. *Finally, if* $G$ *is a topological group and* $p : \mathcal{E} \to \Lambda$ *is a* $G$*-normed vector bundle such that the splitting* $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, *the functional* $J$ *and the Finsler structure* $\|\cdot\|$ *are preserved, i.e.*

$$\left.\begin{array}{l} J(g(\lambda, x)) = J(\lambda, x), \quad \|gx\|_{g\lambda} = \|x\|_\lambda \\ \textit{and} \quad gx \in \mathcal{E}^+ \ (\textit{resp.}\ gx \in \mathcal{E}^-) \end{array}\right\} \tag{A.10}$$

*for any* $g \in G$ *and* $(\lambda, x) \in \mathcal{E}^+$ *(resp.* $\mathcal{E}^-$*), then the above homoeomorphism* $\phi$ *is* $G$*-equivariant, i.e.*

$$\phi(g(\lambda, x+y)) = (g\lambda, \phi_{g\lambda}(gx+gy)) = (g\lambda, g\phi_\lambda(x+y)) = g\phi(\lambda, x+y)$$

*for any* $g \in G$ *and* $(\lambda, x+y) \in \mathcal{E}^+ \oplus \mathcal{E}^-$.

*Proof.* We only need to consider the case $0 < \mathrm{rank}(\mathcal{E}^+) < \infty$. The key is the first two steps corresponding with the proof of Theorem A.1. We can slightly modify the proof of [19, Lemma 2.1] to prove:

**Claim A.3.** *There exists a positive real number* $\epsilon_1 < \delta$ *having the following property: For each* $(\lambda, x) \in B_{\epsilon_1}(\mathcal{E}^+)$ *there exists a unique* $\varphi_\lambda(x) \in B_\delta(\mathcal{E}^-)_\lambda$ *such that*

$$J(\lambda, x + \varphi_\lambda(x)) = \max\{J(\lambda, x+y) \,|\, y \in B_\delta(\mathcal{E}^-)_\lambda\}. \tag{A.11}$$

In fact, the existence of $\epsilon_1$ can be obtained as follows. Since $\bar{B}_\delta(\mathcal{E}^-)$ is compact, suppose by contradiction that there exists a sequence $\{(\lambda_n, x_n)\}$ in $B_\delta(\mathcal{E}^+)$ such that $(\lambda_n, x_n) \to (\lambda_0, \theta_{\lambda_0})$ and a sequence $\{y_n\} \subset \partial B_\delta(\mathcal{E}^-)_{\lambda_n}$ such that

$$J(\lambda_n, x_n + y_n) > J(\lambda_n, x_n + y) \quad \forall y \in B_\delta(\mathcal{E}^-)_{\lambda_n},\ n = 1, 2, \cdots.$$

We may assume $y_n \to y_0 \in \partial B_\delta(\mathcal{E}^-)_{\lambda_0}$. Then

$$\lim_{n\to\infty} J(\lambda_n, x_n + y_n) = J(\lambda_0, y_0) \quad \text{and} \quad \lim_{n\to\infty} J(\lambda_n, x_n) = J(\lambda_0, \theta_{\lambda_0}).$$

Hence $J(\lambda_0, y_0) \ge J(\lambda_0, \theta_{\lambda_0})$. Moreover, by the mean value theorem and Theorem A.2(iii) there exists a $t_y \in (0,1)$ such that

$$J(\lambda_0, y_0) - J(\lambda_0, \theta_{\lambda_0}) = DJ_{\lambda_0}(t_y \cdot y_0)(y_0) = -\frac{1}{t_y} DJ_{\lambda_0}(t_y \cdot y_0)(-t_y \cdot y_0) < 0.$$

This leads to a contradiction.

The uniqueness of $\varphi_\lambda(x)$ can also be proved by contradiction.

Next, as in Step 2 of the proof of Theorem A.1 above we can show that the map

$$B_{\epsilon_1}(\mathcal{E}^+) \to B_{\epsilon_1}(\mathcal{E}^-),\ (\lambda, x) \mapsto (\lambda, \varphi_\lambda(x))$$

is continuous. As in Step 4 above, for $(\lambda, x+y) \in B_{\epsilon_1}(\mathcal{E}^+) \oplus B_\delta(\mathcal{E}^-)$ we define

$$\psi_1(\lambda, x+y) = \begin{cases} \frac{\sqrt{J(\lambda, x+\varphi_\lambda(x))}}{\|x\|_\lambda} x & \text{if } x \ne \theta_\lambda, \\ \theta_\lambda & \text{if } x = \theta_\lambda, \end{cases}$$
$$\psi_2(\lambda, x+y) = \begin{cases} \frac{\sqrt{J(\lambda, x+\varphi_\lambda(x)) - J(\lambda, x+y)}}{\|y-\varphi_\lambda(x)\|_\lambda} (y - \varphi_\lambda(x)) & \text{if } y \ne \varphi_\lambda(x), \\ \theta_\lambda & \text{if } y = \varphi_\lambda(x), \end{cases}$$

and

$$\psi(\lambda, x+y) = \psi_1(\lambda, x+y) + \psi_2(\lambda, x+y). \tag{A.12}$$

They are continuous and $\psi(\lambda, \theta_\lambda) = \theta_\lambda$. Let $\tilde{\psi}(\lambda, x+y) = (\lambda, \psi(\lambda, x+y))$. As in Step 6 above there is a positive real number $\epsilon < \epsilon_1$ such that

$$B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^+) \oplus B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^-) \subset \tilde{\psi}\big(B_{2\epsilon}(\mathcal{E}^+) \oplus B_\delta(\mathcal{E}^-)\big).$$

Set

$$U = \big(B_{2\epsilon}(\mathcal{E}^+) \oplus B_\delta(\mathcal{E}^-)\big) \cap \tilde{\psi}^{-1}\left(B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^+) \oplus B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^-)\right)$$

and

$$\begin{aligned} \phi : B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^+) \oplus B_{\sqrt{p(\epsilon)/2)}}(\mathcal{E}^-) &\to U, \\ (\lambda, x+y) \mapsto (\lambda, \phi_\lambda(x+y)) &:= (\lambda, x'+y'), \end{aligned} \tag{A.13}$$

where $(x', y') \in B_{2\epsilon}(\mathcal{E}^+)_\lambda \oplus B_\delta(\mathcal{E}^-)_\lambda$ is a unique point satisfying $x + y = \psi(\lambda, x'+y')$. Except the final claim we leave the remainder arguments to the reader.

As to the final conclusion, since $\|gx\|_{g\lambda} = \|x\|_\lambda$ for any $g \in G$ and $(\lambda, x) \in \mathcal{E}$, for any $\varepsilon > 0$ the sets $B_\varepsilon(\mathcal{E})$, $B_\varepsilon(\mathcal{E}^+)$ and $B_\varepsilon(\mathcal{E}^-)$ are $G$-invariant. For any $g \in G$ and $(\lambda, x) \in B_{\epsilon_1}(\mathcal{E}^+)$, by Claim A.3 there exists a unique $\varphi_{g\lambda}(gx) \in B_\delta(\mathcal{E}^-)_{g\lambda}$ such that

$$J(g\lambda, gx + \varphi_{g\lambda}(gx)) = \max\{J(g\lambda, gx + y) \,|\, y \in B_\delta(\mathcal{E}^-)_{g\lambda}\}. \tag{A.14}$$

Note that $g : B_\delta(\mathcal{E}^-)_\lambda \to B_\delta(\mathcal{E}^-)_{g\lambda},\ x \mapsto gx$ is a homeomorphism. We conclude

$$\begin{aligned}
\max\{J(g\lambda, gx + y) \,|\, y \in B_\delta(\mathcal{E}^-)_{g\lambda}\} &= \max\{J(g\lambda, gx + gy) \,|\, y \in B_\delta(\mathcal{E}^-)_\lambda\} \\
&= \max\{J(\lambda, x + y) \,|\, y \in B_\delta(\mathcal{E}^-)_\lambda\} \\
&= J(\lambda, x + \varphi_\lambda(x)) \\
&= J(g\lambda, gx + g\varphi_\lambda(x)),
\end{aligned}$$

where the third equality comes from (A.11). Since $g\varphi_\lambda(x) \in B_\delta(\mathcal{E}^-)_{g\lambda}$ it follows from this, (A.14) and Claim A.3 that

$$\varphi_{g\lambda}(gx) = g\varphi_\lambda(x) \quad \forall g \in G \text{ and } (\lambda, x) \in B_{\epsilon_1}(\mathcal{E}^+).$$

Then the desired conclusion follows from this and (A.12)-(A.13). □

# B A few of results on functional analysis

Perhaps the results in this appendix can be founded in some references. For the readers's convenience we shall give proofs of them. Let $E_1$ and $E_2$ be two real normed linear spaces and let $T$ be a map from an open subset $U$ of $E_1$ to $E_2$. For a positive integer $n$ we call $T$ *finite $n$-continuous* at $x \in U$ if for any $h_1, \cdots, h_n \in E_1$ the map

$$\mathbb{R}^n \supseteq B^n(0, \epsilon) \ni t = (t_1, \cdots, t_n) \mapsto T(x + t_1h_1 + \cdots + t_nh_n)$$

is continuous at the origin $0 \in \mathbb{R}^n$.

**Proposition B.1.** **(i)** *If for any $u \in E_1$ the map $x \mapsto DT(x, u)$ is finite 2-continuous at $x_0 \in U$ then $u \mapsto DT(x_0, u)$ is additive.*

**(ii)** *If $T$ is continuously directional differentiable on $U$ then it is strictly $H$-differentiable at every $x \in U$, and restricts to a $C^1$-map on any finitely dimensional subspace. (So the continuously directional differentiability is a notion between the strict $H$-differentiability and $C^1$.)*

**(iii)** *If $T : U \to E_2$ is $G$-differentiable near $x_0 \in U$ and also strictly $G$–differentiable at $x_0$, then $T'$ is strongly continuous at $x_0$, i.e. for any $v \in E_1$ it holds that $\|T'(x)v - T'(x_0)v\| \to 0$ as $\|x - x_0\| \to 0$. In particular, if $E_2 = \mathbb{R}$ this means that $T'$ is continuous with respect to the weak* topology on $E_1^*$.*

*Proof.* **(i)** This directly follows from the mean value theorem. In fact, for $u, v \in E_1$ and a small $t \neq 0$ let $\triangle^2_{tu,tv}T(x_0) = T(x_0+tu+tv) - T(x_0+tu) - T(x_0+tv) + T(x_0)$. Then

$$\lim_{t\to 0} \frac{1}{t} \triangle^2_{tu,tv}T(x_0) = DT(x_0, u+v) - DT(x_0, u) - DT(x_0, v).$$

By the Hahn-Banach theorem there exists a functional $y^* \in E_2^*$ such that $\|y^*\| = 1$ and $y^*(\triangle^2_{tu,tv}T(x_0)) = \|\triangle^2_{tu,tv}T(x_0)\|$. Applying twice the mean value theorem yields $\tau_1, \tau_2 \in [0, t]$ such that

$$\begin{aligned} & y^*(T(x_0+tu+tv) - T(x_0+tu) - T(x_0+tv) + T(x_0)) \\ =\, & y^*(DT(x_0+tv+\tau_1 u, u))t - y^*(DT(x_0+\tau_2 u, u))t \\ \leq\, & \|DT(x_0+tv+\tau_1 u, u) - DT(x_0, u)\| \cdot |t| \\ +\, & \quad \|DT(x_0+\tau_2 u, u) - DT(x_0, u)\| \cdot |t|. \end{aligned}$$

Since the map $x \mapsto DT(x, u)$ is finite 2-continuous at $x_0 \in U$ it follows that

$$\lim_{t\to 0} y^*\Big(\frac{1}{t} \triangle^2_{tu,tv}T(x_0)\Big) = 0.$$

Hence $DT(x_0, u+v) = DT(x_0, u) + DT(x_0, v)$.

**(ii)** Firstly, it follows from (i) that $T$ is Gâteaux differentiable at every $x \in U$ if $T$ is continuously directional differentiable on $U$.

Next we prove that $T$ is strictly $G$-differentiable at every $x \in U$. Otherwise, there exist $x_0 \in U$, $v \in E_1$, $\varepsilon_0 > 0$ and sequences $\{x_n\} \subset U$ with $x_n \to x_0$, $\{t_n\} \subset \mathbb{R} \setminus \{0\}$ with $t_n \to 0$, such that

$$\left\| \frac{T(x_n + t_n v) - T(x_n)}{t_n} - T'(x_0)v \right\| \geq \varepsilon_0 \quad \forall n = 1, 2, \cdots.$$

As above we may use the Hahn-Banach theorem to get a sequence of functionals $y_n^* \in E_2^*$ such that $\|y_n^*\| = 1$ and

$$y_n^* \left( \frac{T(x_n + t_n v) - T(x_n)}{t_n} - T'(x_0)v \right) = \left\| \frac{T(x_n + t_n v) - T(x_n)}{t_n} - T'(x_0)v \right\|$$

for any $n \in \mathbb{N}$. Then the mean value theorem yields a sequence $\{\tau_n\} \subset (0, 1)$ such that

$$y_n^* \left( \frac{T(x_n + t_n v) - T(x_n)}{t_n} - T'(x_0)v \right) = y_n^* \left(T'(x_n + \tau_n t_n v)v - T'(x_0)v\right)$$

$\forall n \in \mathbb{N}$. It follows that

$$\|T'(x_n + \tau_n t_n v)v - T'(x_0)v\| \geq \varepsilon_0 \; \forall n = 1, 2, \cdots.$$

This contradicts to the continuously directional differentiability of $T$.

Finally, suppose that $T$ is not strictly $H$-differentiable at some $x_0 \in U$. Then there exist a compact subset $K \subset E_1$, $\varepsilon_0 > 0$, and and sequences $\{x_n\} \subset U$ with $x_n \to x_0$, $\{t_n\} \subset \mathbb{R} \setminus \{0\}$ with $t_n \to 0$, such that for some sequence $\{v_n\} \subset K$,

$$\left\| \frac{T(x_n + t_n v_n) - T(x_n)}{t_n} - T'(x_0)v_n \right\| \geq \varepsilon_0 \quad \forall n = 1, 2, \cdots .$$

Since $K$ is compact we may assume $v_n \to v_0 \in K$. As just we have a sequence $\{s_n\} \subset (0,1)$ such that $\|T'(x_n + s_n t_n v_n)v - T'(x_0)v_n\| \geq \varepsilon_0$ for all $n \in \mathbb{N}$, which leads to a contradiction.

The second claim can be derived from the fact that the strong convergence and weak one are equivalent on finitely dimensional spaces.

**(iii)** Since $T$ is strictly $G$–differentiable at $x_0$, for any $v \in E_1$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \frac{T(x + tv) - T(x)}{t} - T'(x_0)v \right\| < \varepsilon$$

for any $t \in (-\delta, \delta) \setminus \{0\}$ and $x \in B_X(x_0, \delta)$. Setting $t \to 0$ we get $\|T'(x)v - T'(x_0)v\| \leq \varepsilon \; \forall x \in B_X(x_0, \delta)$. □

**Proposition B.2.** *Suppose that a bounded linear self-adjoint operator $B$ on a Hilbert space $H$ has a decomposition $B = P + Q$, where $Q \in L_s(H)$ is compact and $P \in L_s(H)$ is positive, i.e., $\exists\, C_0 > 0$ such that $(Pu, u)_H \geq C_0\|u\|^2 \; \forall u \in H$. Then every $\lambda \in (-\infty, C_0)$ is either a regular value of $B$ or an isolated point of $\sigma(B)$, which is also an eigenvalue of finite multiplicity.*

*Proof.* Since $(Pu - \lambda u, u)_H = (Pu, u)_H - \lambda\|u\|^2 \geq (C_0 - \lambda)\|u\|^2$ for any $\lambda \in (-\infty, C_0)$ and $u \in H$, it follows from Theorem 9.1-2 in [28] that every $\lambda \in (-\infty, C_0)$ belongs to $\rho(P)$. For such a $\lambda \in (-\infty, C_0)$, observe that

$$\lambda I_H - B = (\lambda I_H - P)[I_H - (\lambda I_H - P)^{-1}Q].$$

So $\lambda I_H - B$ is Fredholm, and hence $\dim \mathrm{Ker}(\lambda I_H - B) < \infty$, $\mathrm{codimKer}(\lambda I_H - B) < \infty$, and $R(\lambda I_H - B) \subset H$ is closed. By Theorem 4.5 on the page 150 of [41], either $\lambda \notin \sigma(B)$ or $\lambda$ is an isolated point of $\sigma(B)$. Clearly, in the latter case $\lambda$ is also an eigenvalue of $B$ with finite multiplicity. □

Actually, this result may also follow from Proposition B.3 below.

By Proposition 4.5 of [16], if $A$ is a continuous linear normal operator (i.e. $A^*A = AA^*$) on a Hilbert space $H$, then for $\lambda \in \sigma(A)$ the range $R(A - \lambda I)$ is closed if and only if $\lambda$ is not a limit point of $\sigma(A)$. As a consequence we deduce that (i) and (ii) of the following proposition are equivalent.

**Proposition B.3.** *Let $H$ be a Hilbert space and let $A \in L(H)$ be a normal operator (i.e. $A^*A = AA^*$). Then the following three claims are equivalent.*

**(i)** $0$ *is at most an isolated point of* $\sigma(A)$*;*

**(ii)** *The range* $R(A)$ *is closed in* $H$*;*

**(iii)** *The operator* $A|_W : W \to W$ *is invertible and its inverse operator* $(A|_W)^{-1} : W \to W$ *is bounded, where* $W = (\mathrm{Ker}(A))^\perp$.

By the Banach inverse operator theorem we arrive at (ii)⇒ (iii). Conversely, $R(A) = A(W) = W$ is closed.

# References


[1] A. Abbondandolo, M. Schwarz, A smooth pseudo-gradient for the Lagrangian action functional. *Adv. Nonlinear Stud.*, **9**(2009), 597-623.

[2] T. Bartsch, Critical Point Theory on Partially Ordered Hilbert Spaces, *J. Funct. Anal.*, **186**(2001), 117C152.

[3] T. Bartsch, K.-C. Chang, Z.-Q. Wang, On the Morse indices of sign changing solutions of nonlinear elliptic problems, *Math. Z.* **233**(2000), 655C677.

[4] T. Bartsch, A. Szulkin, M. Willem, Morse Theory and Nonlinear Differential Equations, *Handbook of global analysis*, 41–73, 2008 - Elsevier Science Ltd.

[5] P. Benevieri and M. Furi, A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree, *Ann. Sci. Math. Quebec* **22**(1998), 131-148.

[6] Benevieri and M. Furi, On the concept of orientability for Fredholm maps between real Banach manifolds, *Topol. Methods Nonlinear Anal.* **16**(2000), 279-306.

[7] M. Berger, *Nonlinearity and Functional Analysis*, Acad. Press (1997).

[8] H.Brézis, L. Nirenberg, $H^1$ versus $C^1$ local minimizers, *C. R. Acad. Sci. Paris Sér. I Math.*, **317**(1993), no. 5, 465-472.

[9] F.E. Browder, Fixed point theory and nonlinear problem, *Bull. Amer. Math. Soc. (N.S)*, **9**(1983), 1-39.

[10] K. C. Chang, *Infinite Dimensional Morse Theory and its applications*, Univ. de Montreal **97** (1985).

[11] K. C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problem*, Birkhäuser, 1993.

[12] K. C. Chang, *Methods in Nonlinear Analysis*, Springer Monogaphs in Mathematics, Springer 2005.

[13] K. C. Chang, $H^1$ versus $C^1$ isolated critical points, *C. R. Acad. Sci. Paris Sér. I Math.*, **319**(1994), no. 5, 441-446.

[14] S. Cingolani and M. Degiovanni, On the Poincar-Hopf theorem for functionals defined on Banach spaces, *Adv. Nonlinear Stud.* **9**(2009), no. 4, 679–699.

[15] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[16] J. B. Conway, *A Course in Functional Analysis*, Springer, New York, 1990.

[17] J. N. Corvellec, Morse theory for continuous functionals, *J. Math. Anal. Appl.* **196**(1995), 1050-1072.

[18] J. Dieudonné, *Fondements de l'Analyse moderne*, Gauthier-Villars (1963).

[19] D. M. Duc, T. V. Hung, N. T. Khai, Morse-Palais lemma for nonsmooth functionals on normed spaces, *Proc. A. M. S.* 135(2007), no. 3., 921-927.

[20] D. M. Duc, T. V. Hung, N. T. Khai, Critical points of non-$C^2$ functionals, *Topological Methods in Nonlinear Analysis*, 29(2007), 35-68.

[21] P.M. Fitzpatrick, J. Pejsachowicz and P.J. Rabier, Orientability of Fredholm Families and Topological Degree for Orientable Nonlinear Fredholm Mappings, *J. Funct. Anal.* **124**(1994), 1-39.

[22] N. Ghoussoub, D. Preiss, A general mountain pass principle for locating and classifying critical points, *A.I.H.P. Analyse non linéaire*, **6**(1989), no.5, p. 321-330.

[23] D. Gromoll, W. Meyer, On differentiable functions with isolated critical points, *Topology* **8**(1969), 361-369.

[24] H. Hofer, The topological degree at a critical point of mountain pass type, *Proc. Symp. Pure Math.*, **45**(1986), 501-509.

[25] D. Husemoller, *Fibre Bundle*, Springer-Verlag 1975.

[26] A. Ioffe, E. Schwartzman, Parametric Morse lemmas for $C^{1,1}$-functions, (English summary) *Recent developments in optimization theory and nonlinear analysis* (*Jerusalem*, 1995), 139C147, *Contemp. Math.*, **204**, *Amer. Math. Soc., Providence, RI*, 1997.

[27] M. Jiang, A generalization of Morse lemma and its applications, *Nonlinear Analysis*, **36**(1999), 943-960.

[28] E. Kreyszig, *Introduction Functional Analysis with Applications*, John wiley & Sons. Ins. 1978.

[29] S. Lang, *Differential Manifolds*, Springer-Verlag, 1985.

[30] C. Li, S. -J. Li, J. Liu, Splitting theorem, Poincare-Hopf theorem and jumping nonlinear problems, *J. Funct. Anal.*, **221**(2005), 439-455.

[31] C. Li, S.-J. Li, and Z. Liu, J. Pan, On the Fucík spectrum, *J. Differential Equations*, **244**(2008), 2498- 2528.

[32] G. Lu, Corrigendum to "The Conley conjecture for Hamiltonian systems on the cotangent bundle and its analogue for Lagrangian systems" [J. Funct. Anal. 256(9)(2009)2967-3034], *J. Funct. Anal.* **261**(2011), 542-589.

[33] G. Lu, The splitting lemmas for nonsmooth functionals on Hilbert spaces. arXiv:1102.2062v1.

[34] G. Lu, Some critical point theorems and applications. arXiv:1102.2136v1.

[35] G. Lu, Methods of infinite dimensional Morse theory for geodesics on Finsler manifolds. *In pregrass*.

[36] Jean Mawhin and Michel Willem, On the generalized Morse Lemma, *Bull. Soc. Math.*, Belgique (B) **37**(1985), 23-29.

[37] Jean Mawhin and Michel Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences Vol.74., Springer-Verlag, 1989.

[38] A. A. Moura, F. M. de Souza, A Morse lemma for degenerate critical points with low differentiabilityhindawi, *Abstract and Applied Analysis*, **5**(2000), no.2, 113-118.

[39] J. Pejsachowicz and P.R. Rabier, Degree theory for $C^1$ Fredholm mappings of index $0$, *J. Anal. Math.* **76**(1998), 289-319.

[40] K. Perera, R. P. Agarwal, Donal O'Regan, *Morse Theoretic Aspects of $p$-Laplacian Type Operators*, Mathematical Surveys and Monographs Vol.161, American Mathematical Society, Providence Rhode Island 2010.

[41] M. Schwartz, *Principles of Functional Analysis*, Acad. Press, 1971.

[42] W. Schirotzek, *Nonsmooth Analysis*, Springer 2007.

[43] I. V. Skrypnik, *Nonlinear Elliptic Equations of a Higher Order* [in Russian], Naukova Dumka, Kiev (1973).

[44] I. V. Skrypnik, *Nonlinear Elliptic Boundary Value Problems*, Teubner, Leipzig, 1986.

[45] A. Tromba, A sufficient condition for a critical point of a functional to be a minimum and its application to Plateau's problem, *Math. Ann.*, **263**(1983), 303-312.

[46] S. A. Vakhrameev, Critical point theory for smooth functions on Hilbert manifolds with singularities and its application to some optimal control problems, *J. Sov. Math.*, **67**(1993), No. 1, 2713-2811.